E E 2 7 6 October 14, 2003

LAB 4- Orthogonal Transforms and Decorrelation Efficiency

Application of one- or two-dimensional transforms to a vector or an array of image elements can reduce the corresponding correlation and thereby allow data compression. Image data typically has strong inter-element correlation in more than one direction. A one-dimensional transform takes into account correlation between

1. Determination of autocorrelation of images

Assuming ergodicity, compute the autocorrelation matrix of the following image:

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

A 3×3 image of the form

$$\begin{bmatrix} g(11) & g(12) & g(13) \\ g(21) & g(22) & g(23) \\ g(31) & g(32) & g(33) \end{bmatrix}$$

To evaluate its autocorrelation function, we write it as a column vector by concatenating columns:

$$\mathbf{g} = [g(11)\ g(21)\ g(31)\ g(12)\ g(22)\ g(32)\ g(13)\ g(23) g(33)]$$

The autocorrelation matrix is then given by $\mathrm{R} = E\{\mathbf{g}\mathbf{g}^T\}$. Instead of averaging over all locations (such as first pixel with last pixel), we can average over all pairs of pixels at the same relative position in the image (since ergodicity is assumed). Thus, the autocorrelation matrix will have the following structure:

	$g11$	$g21$	$g31$	$g12$	$g22$	$g32$	$g13$	$g23$	$g33$
$g11$	A	B	C	D	E	F	G	H	I
$g21$	B	A	B	J	D	E	K	G	H
$g31$	C	B	A	L	J	D	M	K	G
$g12$	D	J	L	A	B	C	D	E	F
$g22$	E	D	J	B	A	B	J	D	E
$g32$	F	E	D	C	B	A	L	J	D
$g13$	G	K	M	D	J	L	A	B	C
$g23$	H	G	K	E	D	J	B	A	B
$g33$	I	H	G	F	E	D	C	B	A

The top row and the left-most column of the matrix show which elements of the image are associated with which in order to produce the corresponding entry in the matrix. A is the average squared element:

$$\frac{\sum_{ij} g_{ij}^2}{9} = \frac{6 \times 1^2 + 3 \times 2^2}{9} = \frac{18}{9} = 2$$

B is the average value of the product of vertical neighbors. We have 6 such pairs. We must sum the product of their values and divide. The question is whether we must divide by the actual number of such pairs (6) or the total number (9). If we divide by the actual number of pairs, then the correlation for certain neighbors such as the most distant ones, which are very few in numbers, will be exaggerated. Hence, we chose to divide by the total number of pixels in the image, acknowledging that this dilutes the contribution of distant pairs, even though this may be significant. (The problem is more significant for small images for which border effects are exaggerated).

$$B = \tfrac{4\times1+2\times4}{9} = \tfrac{12}{9} = 1.33 \qquad \begin{array}{ccc} X & X & X \\ X & X & X \\ X & X & X \end{array}$$

C is the average product of vertical neighbors once removed. We have 3 such pairs:

$$C = \tfrac{2\times1+4}{9} = \tfrac{6}{9} = 0.67 \qquad \begin{array}{ccc} X & X & X \\ X & X & X \\ X & X & X \end{array}$$

D is the average product of horizontal neighbors. There are 6 such pairs:

$$D = \tfrac{6\times2}{9} = \tfrac{12}{9} = 1.33 \qquad \begin{array}{ccc} X & X & X \\ X & X & X \\ X & X & X \end{array}$$

E is the average product of diagonal neighbors. There are 4 such pairs:

$$E = \tfrac{4\times2}{9} = \tfrac{8}{9} = 0.89 \qquad \begin{array}{ccc} X & X & X \\ X & X & X \\ X & X & X \end{array}$$

$$F = \frac{2\times 2}{9} = \frac{4}{9} = 0.44 \qquad \begin{matrix} X & X & X \\ X & X & X \\ X & X & X \end{matrix}$$

$$G = \frac{3\times 1}{9} = \frac{3}{9} = 0.33 \qquad \begin{matrix} X & X & X \\ X & X & X \\ X & X & X \end{matrix}$$

$$H = \frac{2\times 1}{9} = \frac{2}{9} = 0.22 \qquad \begin{matrix} X & X & X \\ X & X & X \\ X & X & X \end{matrix}$$

$$I = \frac{1\times 1}{9} = \frac{1}{9} = 0.11 \qquad \begin{matrix} X & X & X \\ X & X & X \\ X & X & X \end{matrix}$$

$$J = \frac{4\times 2}{9} = \frac{8}{9} = 0.89 \qquad \begin{matrix} X & X & X \\ X & X & X \\ X & X & X \end{matrix}$$

$$K = \frac{2\times 1}{9} = \frac{2}{9} = 0.22 \qquad \begin{matrix} X & X & X \\ X & X & X \\ X & X & X \end{matrix}$$

$$L = \frac{2\times 2}{9} = \frac{2}{9} = 0.44 \qquad \begin{matrix} X & X & X \\ X & X & X \\ X & X & X \end{matrix}$$

$$M = \frac{1 \times 1}{9} = \frac{1}{9} = 0.11 \qquad \begin{matrix} X & X & X \\ X & X & X \\ X & X & X \end{matrix}$$

So, the autocorrelation matrix is:

$$\begin{pmatrix} 2 & 1.33 & 0.67 & 1.33 & 0.89 & 0.44 & 0.33 & 0.22 & 0.11 \\ 1.33 & 2 & 1.33 & 0.89 & 1.33 & 0.89 & 0.22 & 0.33 & 0.22 \\ 0.67 & 1.33 & 2 & 0.44 & 0.89 & 1.33 & 0.11 & 0.22 & 0.22 \\ 1.33 & 0.89 & 0.44 & 2 & 1.33 & 0.67 & 1.33 & 0.89 & 0.44 \\ 0.89 & 1.33 & 0.89 & 1.33 & 2 & 1.33 & 0.89 & 1.33 & 0.89 \\ 0.44 & 0.89 & 1.33 & 0.67 & 1.33 & 2 & 0.44 & 0.89 & 1.33 \\ 0.33 & 0.22 & 0.11 & 1.33 & 0.89 & 0.44 & 2 & 1.33 & 0.67 \\ 0.22 & 0.33 & 0.22 & 0.89 & 1.33 & 0.89 & 1.33 & 2 & 1.33 \\ 0.11 & 0.22 & 0.33 & 0.44 & 0.89 & 1.33 & 0.67 & 1.33 & 2 \end{pmatrix}$$

Experiment:

The following ensemble of images is given:

$$\begin{pmatrix} 5 & 4 & 6 & 2 \\ 5 & 3 & 4 & 3 \\ 6 & 6 & 7 & 1 \\ 5 & 4 & 2 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 5 & 4 & 6 & 2 \\ 5 & 3 & 4 & 3 \\ 6 & 6 & 7 & 1 \\ 5 & 4 & 2 & 3 \end{pmatrix}$$

(1.) Is this ensemble of images erodic with respect to the mean?

(2.) Is this ensemble ergodic with respect to autocorrelation?

{ Hint: Calculate one element of the autocorrelation matrix, say $E\{g(23)g(24)\}$ which is the average of product values of all pixels at position (2,3) and (3,4) over all images.
Compare this with the autocorrelation function which expresses the spatial average of all pixels which are diagonal neighbors from top left to bottom right direction. As per ergodicity, you can look at any one particular image. (Or you can look at many images and average).}

Reference: Chapter 3, Image Processing - The Fundamentals
M. Petrou, P. Bosdogianni, J. Wiley & Sons, 1999