The Wiener Estimator

The Wiener filter is the classic linear noise reduction filter. We consider 1-D here and 2-D in Ch.16 (Castleman).

Suppose we have observed signal $x(t)$ consisting of desired signal $s(t)$ contaminated with additive noise $n(t)$. Want filter to "estimate" the uncontaminated signal.

Partial knowledge We first decide what we expect to know about $s(t)$ and $n(t)$. For our purposes here, we assume $s(t)$ and $n(t)$ ergodic RVs with known power spectra. We also assume that we know the power spectra *a priori* or we can capture samples of $s(t)$ and $n(t)$ and determine their power spectra.

Optimality criteria

We use the *mean square error* criteria. We define the optimal filter as that which minimizes the MSE given by

$$MSE = E\{e^2(t)\} = \int_{-\infty}^{+\infty} e^2(t)dt$$

The latter equality holds because the error signal, being a combination of ergodic variables, is itself a ergodic variable.

About MSE Squaring causes large errors to be "penalized" more severely than small errors.

The Mean Square Error

MSE is a *functional* of $h(t)$. Functional minimization is *covered under calculus of variations* which we employ here. We

- obtain a functional expression of MSE in terms of $h(t)$
- find expression for optimal (minimizing) response $h_o(t)$
- find the corresponding min. MSE with $h_o(t)$ so we know how well filter works.

$$MSE = E\{e^2(t)\} = E\{[s(t) - y(t)]^2\} = E\{s^2(t) - 2s(t)y(t) + y^2(t)\}$$

$$E\{s^2(t) - 2E\{s(t)y(t)\} + E\{y^2(t)\} = T_1 + T_2 + T_3$$

We see that

$$T_1 = \int_{-\infty}^{+\infty} s^2(t)dt = R_s(0)$$

This is known since we assume we know the autocorrelation function of $s(t)$.

$$T_2 = -2E\{s(t)\int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau\}$$

Or equivalently,

$$T_2 = -2\int_{-\infty}^{+\infty} h(\tau)E\{s(t)x(t-\tau)d\tau\} = -2\int$$

We can expand T_3 as the product of two convolutions:

$$T_3 = E\{\int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau \int_{-\infty}^{+\infty} h(\alpha)x(t-\alpha)d\alpha\}$$

which we can arrange as

$$T_3 = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h(\tau)h(\alpha)E\{x(t-\tau)x(t-\alpha)d\tau d\alpha$$

Substituting v = t - α we get $E\{x(t-\tau)x(t-\alpha)\} = E\{x(v-\alpha-\tau)x(v)\}$ which is $R_x(\alpha-\tau)$.
Hence we can write

$$T_3 = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h(\tau)h(\alpha)R_x(\alpha-\tau)d\tau d\alpha$$

Hence we can write the MSE as

$$MSE = R_x(0) - 2\int_{-\infty}^{+\infty} h(\tau)R_{xs}(\tau)d\tau + \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h(\tau)h(\alpha)R_x(\alpha-\tau)d\tau d\alpha$$

This is the MSE in terms of the $h(t), R_x(\tau), R_{xs}(\tau)$. As expected, it is a function of $h(t)$. We wish to select that $h_o(t)$ that causes min MSE.

Ref: Digital Image Processing, Castleman, Sec.11.5.2, Prentice Hall