Paraproducts and the exponential-square class

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Abstract.

We prove that if a locally integrable f has a pointwise bounded dyadic square function, where the square function is defined with respect to a so-called "accretive" weight b, then f is locally exponentially square integrable. We generalize the result to d dimensions by means of "canonical" Haar functions for $L^2(\mathbf{R}^d)$.

In [WhWi], R. L. Wheeden and the author studied weighted norm inequalities for Bergman-type spaces of harmonic functions defined on the upper half-space $\mathbf{R}_+^{d+1}$. For measurable $f : \mathbf{R}^d \mapsto \mathbf{R}$ in some reasonable test class (e.g., L^∞ with bounded support), they obtained strong sufficient conditions on measures μ and non-negative $v \in L^1_{loc}(\mathbf{R}^d)$ which ensured that

$$\left(\int_{\mathbf{R}_+^{d+1}} |\nabla u|^q \, d\mu(x,y)\right)^{1/q} \leq \left(\int_{\mathbf{R}^d} |f|^p \, v \, dx\right)^{1/p} \qquad (*)$$

held for all such f, where, as usual, $u(x,y)$ denotes $P_y * f$, f's Poisson integral, and ∇u is the full gradient.

They worked by considering one derivative at a time, and then looking at a dual form of $(*)$; which, after a little juggling ([WhWi], p. 931), turned out to be a weighted Littlewood-Paley inequality.

Their main tool for attacking the Littlewood-Paley problem was a well-known result from [CWW]: If (one of many variants of) the Lusin square function of f is bounded, then f belongs locally to the exponential L^2 class. The fundamental result in [CWW] was that for the square function based on the Haar system, which we now define.

Let $\mathcal{D}$ denote the family of dyadic intervals on $\mathbf{R}$. If $I \in \mathcal{D}$, we let I_l and I_r denote I's respective left and right halves, and we set:

$$h_{(I)} = \begin{cases} \frac{1}{\sqrt{|I|}} & \text{if } x \in I_l; \\ \frac{-1}{\sqrt{|I|}} & \text{if } x \in I_r; \\ 0 & \text{otherwise,} \end{cases}$$

where we mean $|I|$ to denote I's Lebesgue measure.

For any $f \in L^1_{loc}(\mathbf{R})$, we define the *dyadic square function* $S_d(f)$ by:

$$S_d(f)(x) = \left(\sum_{I \in \mathcal{D}} \frac{|\langle f, h_{(I)}\rangle|^2}{|I|} \chi_I(x)\right)^{1/2}.$$

The precise exponential L^2 result from [CWW], and the starting point for [WhWi], is this: *If* $\|S(f)\|_\infty \leq 1$ *then, for all* $\lambda > 0$ *and all dyadic intervals* I,

$$|\{x \in I : |f(x) - f_I| > \lambda\}| \leq 2|I| \exp(-\lambda^2/2),$$

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where f_I denotes f's average over I.

In trying to generalize the results of [WhWi] to non-smooth (Lipschitz) domains, one quickly encounters a problem. The analogue of $(*)$ dualizes to a weighted Littlewood-Paley inequality, and there is, at least in $d = 1$, an appropriate dyadic square function ready to hand (see below). But it isn't clear that this square function satisfies the analogous exponential L^2 estimate, because the martingale proof from [CWW] doesn't work for it.

Proving such an estimate (which we believe to be of independent interest) is the purpose of this note. (The applications to Bergman-type inequalities on Lipschitz domains, which involve many more technicalities, will appear elsewhere [W2].)

We begin by defining the appropriate square function.

In 1989, in connection with their work on the Cauchy integral operator on a Lipschitz domain, Coifman, Jones, and Semmes [CJS] introduced "Haar functions" with respect to certain complex-valued functions (weights) b. Let us say that $b \in L^\infty(\mathbf{R})$ is *accretive* if there is a $\delta > 0$ such that $|b_I| > \delta$ for all $I \in \mathcal{D}$. If b is accretive then, for each $I \in \mathcal{D}$, we may define

$$h_{(I),b} = \frac{1}{\sqrt{b(I)}} \left[\frac{\sqrt{b(I_l)}}{\sqrt{b(I_r)}} \chi_{I_r} - \frac{\sqrt{b(I_r)}}{\sqrt{b(I_l)}} \chi_{I_l} \right],$$

for some fixed choice of the square roots, and where we have set $b(E) \equiv \int_E b$. The family $\{h_{(I),b}\}_I$ is "orthonormal" with respect to the weight $b(x)$. For any $f \in L^1_{loc}(\mathbf{R})$ we may define b-based "Haar coefficients" $\lambda_{I,b}(f)$ by

$$\lambda_{I,b}(f) = \int f(x)\, h_{(I),b}(x)\, b(x)\, dx.$$

In [CJS] it is proved that, for any $f \in L^2(\mathbf{R})$, the sum $\sum_I \lambda_{I,b}(f) h_{(I),b}$ converges to f in L^2, and that there are positive constants c_1 and c_2, depending only on b, such that

$$c_1 \sum_I |\lambda_{I,b}(f)|^2 \le \|f\|_2^2 \le c_2 \sum_I |\lambda_{I,b}(f)|^2.$$

For any $f \in L^1_{loc}(\mathbf{R})$, and b accretive, we define the b-based dyadic square function $S_{d,b}(f)$ by:

$$S_{d,b}(f)(x) \equiv \left(\sum_{I \in \mathcal{D}} \frac{|\lambda_{I,b}(f)|^2}{|I|} \chi_I(x) \right)^{1/2}.$$

Our main result is:

Theorem 1. *Let b be accretive. There exist positive constants c_1 and c_2, depending only on b, such that, for all $f \in L^1_{loc}(\mathbf{R})$, if $S_{d,b}(f) \le 1$ almost everywhere then, for all $J \in \mathcal{D}$ and all $\lambda > 0$,*

$$|\{x \in J : \; |f(x) - f_J| > \lambda\}| \le c_1 |J| \exp(-c_2 \lambda^2).$$

The theorem will follow from a simple lemma. The statement of this lemma will go more smoothly with the help of three definitions.

Definition 1. *Let $\mathcal{D}_0$ denote the collection of dyadic subintervals of $I_0 \equiv [0,1]$. We say that a family of functions $\{\phi_{(I)}\}_I$, indexed over $\mathcal{D}_0$, is Haar-like if each $\phi_{(I)}$ is supported on I, is constant on I_l and I_r, and if there is a positive constant A such that, for all $I \in \mathcal{D}_0$,*

$$\|\phi_{(I)}\|_\infty \le A|I|^{-1/2}.$$

Definition 2. *We say that a Haar-like family $\{\phi_{(I)}\}_I$ has the subgaussian property if there are positive constants c_1 and c_2 such that, whenever $f = \sum_I \lambda_I \phi_{(I)}$ is a finite linear sum satisfying*

$$\sum_I \frac{|\lambda_I|^2}{|I|}\chi_I(x) \leq 1$$

for almost all $x \in I_0$, we also have, for all $\lambda > 0$,

$$|\{x \in I_0 : \ |f(x)| > \lambda\}| \leq c_1 \exp(-c_2\lambda^2).$$

Definition 3. *We say that a Haar-like family $\{\phi_{(I)}\}_I$ has the Carleson property if there is a positive constant B such that, for all dyadic $J \subset I_0$,*

$$\sum_{I: I\subset J} |\int \phi_{(I)}|^2 \leq B|J|.$$

Lemma 1. *If a Haar-like family $\{\phi_{(I)}\}_I$ has the Carleson property, it has the subgaussian property.*

Proof. Set $c(I) = \int \phi_{(I)}$ and define

$$\psi_{(I)} \equiv \phi_{(I)} - \frac{c(I)}{|I|}\chi_I.$$

The family $\{\psi_{(I)}\}_I$ is Haar-like; indeed, the $\psi_{(I)}$'s are boundedly constant multiples of the ordinary Haar functions. By the martingale proof from [CWW], $\{\psi_{(I)}\}_I$ has the subgaussian property. Set

$$\tilde{f} = \sum_I \frac{|\lambda_I c(I)|}{|I|}\chi_I,$$

where we assume that

$$\sum_I \frac{|\lambda_I|^2}{|I|}\chi_I(x) \leq 1$$

almost everywhere. Let g be non-negative and supported in I_0. Then:

$$\begin{aligned}
\int \tilde{f}\, g\, dx &= \sum_I |\lambda_I||c(I)|\frac{1}{|I|}\int_I g \qquad (1)\\
&\leq \left(\sum_I \frac{|\lambda_I|^2}{|I|}\int_I g\right)^{1/2}\left(\sum_I |c(I)|^2\frac{1}{|I|}\int_I g\right)^{1/2}\\
&= \left(\int [\sum_I \frac{|\lambda_I|^2}{|I|}\chi_I(x)] g\, dx\right)^{1/2}\left(\sum_I |c(I)|^2\frac{1}{|I|}\int_I g\right)^{1/2}\\
&\leq C\left(\int g\right)^{1/2}\left(\int_{I_0} Mg\right)^{1/2}, \qquad (2)
\end{aligned}$$

where the final inequality follows from our assumption on the λ_I's and the Carleson property, and Mg denotes the Hardy-Littlewood maximal function. We recall that

$$\int_{I_0} Mg \leq C\int_{I_0} g(x)\log(e + g(x)/g_{I_0})\, dx$$

(see [St]). Let $E_\lambda = \{x \in I_0 : \tilde{f}(x) > \lambda\}$ and set $g = \chi_{E_\lambda}$. Then the left-hand side of (1) is bounded below by $\lambda|E_\lambda|$, while the right-hand side of (2) is bounded above by $C|E_\lambda|\sqrt{\log(e + 1/|E_\lambda|)}$. Putting them together, we get

$$\lambda^2 \leq C \log(e + 1/|E_\lambda|),$$

which implies the lemma. QED.

Proof of the theorem. Without loss of generality, we may assume that $J = I_0$ and that f is a finite linear sum:

$$f = \sum_I \lambda_I h_{(I),b} = \sum_I \lambda_{I,b}(f) h_{(I),b}.$$

Write $f = f_1 + f_2$, where

$$f_1 = \sum_{I \notin \mathcal{D}_0} \lambda_I h_{(I),b}$$

$$f_2 = \sum_{I \in \mathcal{D}_0} \lambda_I h_{(I),b}.$$

The family $\{h_{(I),b}\}_{I \in \mathcal{D}_0}$ is clearly Haar-like. A computation shows that $|\int h_{(I),b}| \leq C(b)|\int b h_{(I)}|$. It is well-known (and easy to show) that, for any dyadic $J \subset I_0$,

$$\sum_{\substack{I \in \mathcal{D}_0 \\ I \subset J}} |\int b h_{(I)}|^2 \leq \|b\|_\infty^2 |J|.$$

By the lemma, $\{h_{(I),b}\}_{I \in \mathcal{D}_0}$ has the subgaussian property. Therefore, if $\|S_{d,b}(f)\|_\infty \leq 1$, we have

$$|\{x \in I_0 : |f_2(x)| > \lambda\}| \leq c_1 \exp(-c_2 \lambda^2). \tag{3}$$

But $f_2 = f - f_{I_0,b}$ on I_0, where we are setting

$$f_{I_0,b} \equiv \frac{1}{b(I_0)} \int_{I_0} f b \, dx,$$

f's "b-average" over I_0. Inequality (3) now implies that

$$|f_{I_0} - f_{I_0,b}| \leq \int_{I_0} |f - f_{I_0,b}| \, dx \leq C(b),$$

finishing the proof of the theorem. QED.

The astute reader will note that the theorem implies several interesting corollaries, with proofs essentially identical to those that work when $b \equiv 1$. We will state only three of them, interspersed with some explanatory definitions. Let us first set

$$f_b^*(x) \equiv \sup_{\substack{I \in \mathcal{D} \\ x \in I}} |f_{I,b}|,$$

the b-based dyadic maximal function of f.

Corollary 1. *Let $b : \mathbf{R} \mapsto \mathbf{C}$ be accretive. There are positive constants $c_1 = c_1(b)$ and $c_2 = c_2(b)$ such that, for all positive numbers λ and γ,*

$$|\{x : \; f_b^*(x) > 2\lambda, \, S_{d,b}(f)(x) \le \gamma\lambda\}| \le c_1 \exp(-c_2\gamma^{-2})|\{x : \; f_b^*(x) > \lambda\}|.$$

For any number $\eta \ge 0$, and v a non-negative function in $L^1_{loc}(\mathbf{R})$, define

$$M_\eta v(x) = \sup_{I \ni x} \frac{1}{|I|} \int_I v(t) \, \log^\eta(e + v(t)/v_I) \, dt,$$

where the supremum is taken over all dyadic intervals containing x. If $\eta = 0$, this is the usual Hardy-Littlewood maximal function. If $\eta = k$, then M_η shares some properties with a $(k+1)$-fold iteration of M (see [WhWi], pp. 949-953), with the two advantages that M_η can be estimated without an iteration and is smaller. For the un-b version of the next corollary, we refer the reader to ([W1], pp. 668 and 670) and ([WhWi], p. 949).

Corollary 2. *If $0 < p < \infty$ and $\eta > p/2 - 1$, then*

$$\int_{\mathbf{R}} |f_b^*|^p \, v \, dx \le C \int_{\mathbf{R}} S^p_{d,b}(f) \, M_\eta v \, dx,$$

for all $f \in \mathcal{C}_0^\infty(\mathbf{R})$ and all non-negative weights v, with a constant C only depending on p, η, and b. In particular, if $0 < p < 2$, we have

$$\int_{\mathbf{R}} |f_b^*|^p \, v \, dx \le C_{p,b} \int_{\mathbf{R}} S^p_{d,b}(f) \, Mv \, dx.$$

The third corollary is of "Bergman-type" (for the original version, see [WhWi], p. 926). For $1 < p < \infty$ and v a non-negative weight, define $\sigma = v^{1-p'}$, where p' is the dual exponent to p.

Corollary 3. *Let $1 < p \le q < \infty$, with $q \ge 2$. Let $\{\mu_I\}_{I \in \mathcal{D}}$ be a sequence of non-negative numbers indexed over $\mathcal{D}$, and let v be a non-negative weight. In order that*

$$\left(\sum_I |\lambda_{(I,b)}(f)|^q \, \mu_I \right)^{1/q} \le \left(\int_{\mathbf{R}} |f|^p \, v \, dx \right)^{1/p}$$

should hold for all bounded, compactly-supported f, it is sufficient that

$$\sup_{I \in \mathcal{D}} \mu_I^{1/q} \left(\int_I \sigma(t) \, \log^\eta(e + \sigma(t)/\sigma_I) \, dt \right)^{1/p'} |I|^{-1/2} \le C,$$

for some $\eta > p'/2$, and with C depending only on p, q, b, and η.

Theorem 1 and its corollaries will generalize to the setting of $\mathbf{R}^d$ as soon as we have defined the appropriate d-dimensional "Haar functions" and (very slightly) modified our definition of accretivity. Let us now proceed to do so. We shall denote the family of dyadic cubes in $\mathbf{R}^d$ by $\mathcal{D}_d$. For any $Q \in \mathcal{D}_d$, we set $\mathcal{E}(Q) \equiv \{Q' \in \mathcal{D}_d : \; Q' \subset Q, \, \ell(Q') = (1/2)\ell(Q)\}$, the family of 2^d immediate dyadic subcubes of Q. Let us say that $b \in L^\infty(\mathbf{R}^d)$ is *d-accretive* if there is a $\delta > 0$ such that, for all $Q \in \mathcal{D}_d$ and all $E \subset Q$, where E is a nontrivial (and necessarily finite) union from $\mathcal{E}(Q)$, $|b(E)| > \delta|Q|$.

Lemma 2. *Let $Q \in \mathcal{D}_d$. There exist $2^d - 1$ pairs of sets $\{(E(1,j,Q),\, E(2,j,Q))\}_1^{2^d-1}$ such that:*

a) for each j, $E(1,j,Q)$ and $E(2,j,Q)$ are non-empty unions from $\mathcal{E}(Q)$;

b) for each j, $E(1,j,Q) \cap E(2,j,Q) = \emptyset$;

c) for every $j \neq k$, one of the following must hold: i) $E(1,j,Q) \cup E(2,j,Q)$ is entirely contained in either $E(1,k,Q)$ or $E(2,k,Q)$; ii) $E(1,k,Q) \cup E(2,k,Q)$ is entirely contained in either $E(1,j,Q)$ or $E(2,j,Q)$; iii) $[E(1,j,Q) \cup E(2,j,Q)] \cap [E(1,k,Q) \cup E(2,k,Q)] = \emptyset$.

Remark. Once we have the pairs of sets, the definition of the b-based "Haar functions" is immediate. Simply set:

$$h_{j,(Q),b} = \frac{1}{\sqrt{b(E(1,j,Q) \cup E(2,j,Q))}} \left[\frac{\sqrt{b(E(1,j,Q))}}{\sqrt{b(E(2,j,Q))}} \chi_{E(2,j,Q)} - \frac{\sqrt{b(E(2,j,Q))}}{\sqrt{b(E(1,j,Q))}} \chi_{E(1,j,Q)} \right];$$

and, in the special case of $b \equiv 1$, set $h_{j,(Q)} = h_{j,(Q),b}$. The inclusion and disjointness properties of the $E(l,j,Q)$'s ensure that

$$\int h_{j,(Q),b}\, b\, dx = 0,$$

for all j; and, for all j and j',

$$\int h_{j,(Q),b} h_{j',(Q),b}\, b\, dx = \begin{cases} 1 & \text{if } j = j' \\ 0 & \text{if } j \neq j', \end{cases}$$

where the second condition (orthogonality) follows from the first, because either the two Haar functions have disjoint supports, or one will have its support entirely contained in a set across which the other is constant.

Remark. The d-accretivity of b and the simple form of the $h_{j,(Q),b}$'s imply that $|\int h_{j,(Q),b}| \leq C(b)|\int b h_{j,(Q)}|$.

Proof of Lemma 2. Without loss of generality, we take $Q = Q_0 = [0,1)^d$, the dyadic unit cube. The proof is by induction on d. The result is clearly true when $d = 1$. Now assume it's true for $d-1$ and write $Q_0 = S \times [0,1) = \{(x,t) :\ x \in S,\, t \in [0,1)\}$, where S is the $(d-1)$-dimensional unit cube. Let $\{(\tilde{E}(1,j,S),\, \tilde{E}(2,j,S))\}_1^{2^{d-1}-1}$ be (by induction) the corresponding pairs of sets for S. We get $2^d - 2$ of the pairs for Q_0 by taking Cartesian products:

$$\{(\tilde{E}(1,j,S) \times [0,1/2),\, \tilde{E}(2,j,S) \times [0,1/2))\}_1^{2^{d-1}-1} \cup \{(\tilde{E}(1,j,S) \times [1/2,1),\, \tilde{E}(2,j,S) \times [1/2,1))\}_1^{2^{d-1}-1}.$$

The last pair of sets is $(S \times [0,1/2),\, S \times [1/2,1))$. The verification of properties a), b), and c) follows easily from the inductive hypotheses, and is left to reader.

With the $h_{j,(Q)}$'s and $h_{j,(Q),b}$'s in hand, the generalization of Theorem 1 to $\mathbf{R}^d$ is immediate. For $f \in L^1_{loc}(\mathbf{R}^d)$, define

$$\lambda_{j,(Q)}(f) = \int f\, h_{j,(Q)}\, dx$$

$$\lambda_{j,(Q),b}(f) = \int f\, h_{j,(Q),b}\, b(x)\, dx,$$

and corresponding square functions:

$$\tilde{S}_d(f)(x) = \left(\sum_{Q \in \mathcal{D}_d} \frac{|\lambda_{j,(Q)}(f)|^2}{|Q|} \chi_Q(x) \right)^{1/2}$$

$$\tilde{S}_{d,b}(f)(x) = \left(\sum_{Q \in \mathcal{D}_d} \frac{|\lambda_{j,(Q),b}(f)|^2}{|Q|} \chi_Q(x) \right)^{1/2}.$$

The family $\{h_{j,(Q)}\}_{j,Q}$ is easily seen to be orthonormal and complete in $L^2(\mathbf{R}^d)$. Therefore, for all $f \in L^2$,

$$\int_{\mathbf{R}^d} |f|^2\, dx = \sum |\lambda_{j,(Q)}(f)|^2.$$

Note that the $h_{j,(Q)}$'s and $h_{j,(Q),b}$'s are constant on the immediate dyadic subcubes of Q. Now the argument from [CJS], with trivial modifications, implies that for any d-accretive b there are positive constants c_1 and c_2, depending on b and d, such that

$$c_1 \sum |\lambda_{j,(Q),b}(f)|^2 \leq \int_{\mathbf{R}^d} |f|^2\, dx \leq c_2 \sum |\lambda_{j,(Q),b}(f)|^2$$

for all $f \in L^2$.

An almost-verbatim repetition of the proof of Theorem 1 yields:

Theorem 2. *Let $b : \mathbf{R}^d \mapsto \mathbf{C}$ be d-accretive. There exist positive constants c_1 and c_2, depending only on b and d, such that, for all $f \in L^1_{loc}(\mathbf{R}^d)$, if $\tilde{S}_{d,b}(f) \leq 1$ almost everywhere then, for all $Q \in \mathcal{D}_d$ and all $\lambda > 0$,*

$$|\{x \in Q :\ |f(x) - f_Q| > \lambda\}| \leq c_1 |Q| \exp(-c_2 \lambda^2).$$

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