A Kuratowski Theorem for the Projective Plane

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0. INTRODUCTION

A graph G is irreducible for a surface S if G does not embed in S but any proper subgraph of G does embed in S. Kuratowski's theorem states that any graph which is irreducible for the sphere is homeomorphic to either K_5 or to $K_{3,3}$. H. Glover, J. Huneke, and Ç.S. Wang [5] have constructed a list of 103 pair-wise nonhomeomorphic graphs which are irreducible for the projective plane.

Theorem. Their list is complete, i.e., any graph which is irreducible for the projective plane is homeomorphic to a graph in their list.

1. THE MAIN RESULT

An *embedding* of a graph G into a surface S is a realization of G as a subspace of S. A graph G is *irreducible* for S if G does not embed in S, but any proper subgraph of G does embed in S. Irreducible graphs are the smallest (with respect to containment) graphs which fail to embed on a given surface. Let $I(S)$ denote the set of graphs, each with no valency 2 vertices, which are irreducible for S. Using this notation we state Kuratowski's theorem [7]:

$$I(S_0) = \{K_5, K_{3,3}\},$$

where S_0 denotes the 2-dimensional sphere.

It is easy to show that a graph G does not embed in S if and only if G contains a subgraph homeomorphic to a graph in $I(S)$. Thus, determining $I(S)$ gives a characterization of graphs which do not embed in S.

The real projective plane P is defined as the orbit space of the antipodal involution on the 2-sphere. P can also be described as the nonorientable surface of nonorientable genus 1, i.e., the sphere with a single crosscap sewn in. H. Glover, J. Huneke, and C. S. Wang [5] have constructed a list of 103 graphs which are irreducible for the projective plane.

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Theorem 1. Their list is complete, i.e.,

$$|I(P)| = 103.$$

For drawings of the 103 graphs we direct the reader to [5].

We note that Glover and Huneke [3] have shown that the set $I(P)$ is finite. Although no bound was specified in their paper, it is clearly much larger than the number 103 shown here. Also the set of cubic graphs in $I(P)$ was found by Glover and Huneke [4] and independently by Milgram [8].

The real projective plane is the last surface S for which an explicit listing of $I(S)$ seems realistic. The number of irreducible graphs appears to grow quite rapidly as the number of handles, or number of crosscaps, increases. For example, it is straightforward to show that the number of separable graphs which are irreducible for the Klein bottle is over 4000. Partial lists of graphs which are irreducible for the torus have been constructed by Haggard and by Decker [2], but these lists are by no means complete. It is not known if the set $I(S)$ is finite for any surface other than the sphere on the projective plane; although a proof was announced in 1969 it remains unsubstantiated.

We define the *Kuratowski cover number* of G, $K(G)$, as the least n such that

$$G = \bigcup_{i=1}^{n} H_i,$$

where each H_i is homeomorphic to either K_5 or to $K_{3,3}$. It has been conjectured that $G \in I(S)$ implies $K(G) \leq 3 - \chi(S)$, where $\chi(S)$ denotes the Euler characteristic of S. By examining each of the 103 graphs in $I(P)$ we see that this conjecture is true for the projective plane, i.e., if G is irreducible for the projective plane, then G can be written as the union of two subgraphs each homeomorphic to either K_5 or $K_{3,3}$.

2. SKETCH OF THE PROOF

Due to the length of the proof of Theorem 1 we present here only a brief

Let v be a vertex of a graph G and let $1,2,\ldots,j,j+1,\ldots,n$ be the vertices adjacent to v. From G we can construct a new graph, $S_{v:(1,\ldots,j)}(G)$, or more succinctly $S_v(G)$ by deleting the edges $\{(v,k) \mid k = j+1,\ldots,n\}$ and adding a new vertex v' and edges (v, v) and $\{(v', k) \mid k = j+1,\ldots,n\}$. Observe that G can be recovered from $S_v(G)$ by contracting the edge (v, v'). By examining the contrapositive we see:

Lemma. If G does not embed in S then $S_v(G)$ does not embed in S.

Let $G \in I(S)$. Since $S_v(G)$ does not embed in S it must contain a subgraph H homeomorphic to some $G' \in I(S)$. In this case we say G' is an *elementary derivation* of G, denoted $G \geq G'$. As an example $K_{3,3}$ may be constructed from K_5 by splitting a vertex and deleting two edges.

Consider the reflexive transitive relation on $I(S)$, also denoted $\geq$, generated by elementary derivations.

Lemma. For each surface S, $(I(S), \geq)$ is a partially ordered set.

Let $I^M(S)$ denote the set of maximal graphs in $(I(S), \geq)$. Maximal graphs are of interest because by a sequence of vertex splittings and edge deletions we can generate large numbers of irreducible graphs. In fact, using this process the 103 graphs in $I(P)$ were constructed from a set of 5 graphs conjectured to be maximal [6]. Glover, Huneke, and Wang have announced [6] that these 103 graphs are the only irreducible graphs generated by this set of 5 graphs. This is independently verified in [1].

Theorem 1 was proved by first finding the set $I^M(P)$ and then checking that the 103 graphs were the only graphs constructable from this set. In practice a coarser partial ordering was used, making it easier to show that a given graph was maximal and greatly simplifying the derivation of the 103 graphs. For details see [1].

For any surface S the set $I^M(S)$ is of special interest; observe bounding $|I^M(S)|$ also bounds $|I(S)|$. On the other hand let $I_3(S)$ denote the set of cubic graphs in $I(S)$, i.e., the set of minimal graphs in $(I(S), \geq)$. Given an arbitrary graph one could check its genus by splitting (in all possible ways) to a set of cubic graphs and looking for minimal irreducible subgraphs. Characterizing either of these sets for any surface except the sphere and the projective plane remains an open question.

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