Migration of Pebbles over a Landscape

We are going to solve the following problem: *A pebble is resting at a point on a hillside, held in place by friction. As the result of rainfall, or a rise in ground water, the coefficient of friction between the pebble and the ground is suddenly decreased, and the pebble begins to slide. Describe its subsequent trajectory.*

We idealize the problem as follows: The pebble is a particle of mass m, and we neglect any impetus given to it by a raindrop falling upon it. The surface of the ground is represented by an equation $f(x, y) - z = 0$, where (x, y) are coordinates in a fixed horizontal plane and $z = f(x, y)$ is the height of the ground above point (x, y) in the reference plane. We assume that the only forces acting on the particle are the following: (1) its weight mg, where g is the acceleration of gravity; (2) the supporting force exerted by the ground, which acts perpendicularly to the surface of the ground; (3) the frictional force, which acts tangentially to the surface of the ground in the direction opposite to the slide. Since these last two forces are purely reactive, we need to analyze them more closely.

Consider a particle moving along a surface whose equation is $\varphi(x, y, z) = 0$. Let the position of the particle at time t be $\boldsymbol{r}(t) = x(t)\boldsymbol{i} + y(t)\boldsymbol{j} + z(t)\boldsymbol{k}$, so that $\varphi\bigl(\boldsymbol{r}(t)\bigr) = 0$. Differentiating with respect to t, we find

$$\nabla\varphi\bigl(\boldsymbol{r}(t)\bigr) \cdot \boldsymbol{r}'(t) \equiv 0, \tag{1}$$

where $\nabla\varphi = \frac{\partial\varphi}{\partial x}\boldsymbol{i} + \frac{\partial\varphi}{\partial y}\boldsymbol{j} + \frac{\partial\varphi}{\partial z}\boldsymbol{k}$ is the gradient of the surface. This equation merely says that the velocity $\boldsymbol{r}'(t)$ of the particle is perpendicular to the gradient of the function φ, which of course follows, since the gradient is perpendicular to the tangent plane of the surface, and the velocity of the particle is always tangent to the surface. If we differentiate again and multiply by the mass m of the particle, we find the equation

$$\nabla\varphi\bigl(\boldsymbol{r}(t)\bigr) \cdot m\boldsymbol{r}''(t) \equiv -m\Delta\bigl(\boldsymbol{r}(t), \boldsymbol{r}'(t)\bigr), \tag{2}$$

where

$$\begin{aligned} m\Delta\bigl(\boldsymbol{r}(t), \boldsymbol{r}'(t)\bigr) &= m\,[\,x'(t) \quad y'(t) \quad z'(t)\,] \begin{bmatrix} \frac{\partial^2\varphi}{\partial x^2} & \frac{\partial^2\varphi}{\partial x\partial y} & \frac{\partial^2\varphi}{\partial x\partial z} \\ \frac{\partial^2\varphi}{\partial y\partial x} & \frac{\partial^2\varphi}{\partial y^2} & \frac{\partial^2\varphi}{\partial y\partial z} \\ \frac{\partial^2\varphi}{\partial z\partial x} & \frac{\partial^2\varphi}{\partial z\partial y} & \frac{\partial^2\varphi}{\partial z^2} \end{bmatrix} \begin{bmatrix} x'(t) \\ y'(t) \\ z'(t) \end{bmatrix} \\ &= m\Bigl(\frac{\partial^2\varphi}{\partial x^2}\bigl(x'(t)\bigr)^2 + \frac{\partial^2\varphi}{\partial y^2}\bigl(y'(t)\bigr)^2 + \frac{\partial^2\varphi}{\partial z^2}\bigl(z'(t)\bigr)^2\Bigr) \\ &\qquad + 2m\Bigl(\frac{\partial^2\varphi}{\partial x\partial y}x'(t)y'(t) + \frac{\partial^2\varphi}{\partial x\partial z}x'(t)z'(t) + \frac{\partial^2\varphi}{\partial y\partial z}y'(t)z'(t)\Bigr). \end{aligned} \tag{3}$$

The left-hand side of Eq. (2), when multiplied by the vector $\frac{\nabla\varphi}{\nabla\varphi\cdot\nabla\varphi}$, yields the normal component of the force on the particle. *This must be the total normal component of the force acting on the particle if the particle is to move along the surface.* The right-hand side of Eq. (2), that is, the function $-m\Delta$, when multiplied by the same vector, then gives an expression for this force in terms of the curvature of the surface (the first and second derivatives of φ) and the velocity of the particle. In our case this normal force arises from two sources: (1) The normal component of the gravitational force $-mg\boldsymbol{k}$; (2) the resistive force exerted by the ground surface, preventing the particle from sinking into the ground. What we have established is that the force exerted by the surface is the difference between the total normal force given by the two sides of Eq. (2) (after multiplication by the vector mentioned above) and the normal component of the gravitational force. Because we are able to compute the total normal force by using Eq. (2) and we can easily find an expression for the normal component of the gravitational force, it is possible to compute the force exerted on the particle by the surface. It is important to do so, since the frictional force arises as a result of that force alone.

In the present case, where $\varphi(x, y, z) = f(x, y) - z$, we note that the expression $\Delta(x, y, z, x', y', z')$ is independent of z and z' and is given by

$$\Delta(x, y, x', y') = (x')^2 \frac{\partial^2 f}{\partial x^2} + 2x'y' \frac{\partial^2 f}{\partial x \partial y} + (y')^2 \frac{\partial^2 f}{\partial y^2}. \tag{4}$$

We shall abbreviate this expression in what follows, writing it simply as Δ.

It is crucial for our analysis that the force exerted by the ground on the particle is the difference of the net normal force and the normal component of the gravitational attraction, since the magnitude of the frictional force opposing the tangential movement of the particle is simply the coefficient of friction μ times the magnitude of the force exerted by the surface. We shall assume that the normal component of the gravitational force is always larger than the force required to confine the particle to the surface. In other words, we assume that the particle does not leave the surface like a skier going over a mogul. As we shall see below, this requirement means that $\Delta + g > 0$.

The normal vector to the surface is the gradient of $\varphi(x, y, z) = f(x, y) - z$. We write this vector as $\boldsymbol{n}$, so that

$$\boldsymbol{n} = \frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} - \boldsymbol{k}. \tag{5}$$

The magnitude of $\boldsymbol{n}$ is

$$|\boldsymbol{n}| = \sqrt{\boldsymbol{n} \cdot \boldsymbol{n}} = \left(\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1 \right)^{\frac{1}{2}}. \tag{6}$$

Thus the total normal force $\boldsymbol{N}$ on the particle is given by

$$\boldsymbol{N} = \frac{-m\Delta}{\boldsymbol{n} \cdot \boldsymbol{n}}\boldsymbol{n} = \frac{-m\Delta}{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}\boldsymbol{n}. \tag{7}$$

The gravitational force on the particle is $\boldsymbol{G} = -mg\boldsymbol{k}$, and its normal component is

$$\boldsymbol{G}_n = \frac{-mg\boldsymbol{k} \cdot \boldsymbol{n}}{\boldsymbol{n} \cdot \boldsymbol{n}}\boldsymbol{n} = \frac{mg}{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}\boldsymbol{n}. \tag{8}$$

The requirement that the particle never leave the surface is equivalent to the requirement that the direction of the force exerted by the surface on the particle, that is, $\boldsymbol{N} - \boldsymbol{G}_n$, be opposite to the direction of $\boldsymbol{G}_n$. Since $\boldsymbol{N} - \boldsymbol{G}_n$ is a purely resisting force, it does not arise at all unless $\boldsymbol{G}_n$ is so large that it would push the particle under the surface in the absence of the resisting force. By looking at the expressions for these vectors in (7) and (8), we see that $g + \Delta$ must have the same sign as g. Thus, the condition is that $g + \Delta > 0$, as stated above.

The tangential component of the gravitational force is

$$\boldsymbol{G}_t = \boldsymbol{G} - \boldsymbol{G}_n = \frac{-mg}{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1} \left(\frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} + \left(\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 \right)\boldsymbol{k} \right). \tag{9}$$

The frictional force has the direction opposite to $\boldsymbol{r}'$, and its magnitude is μ times the magnitude of the normal force exerted by the surface on the particle, that is, its magnitude is μ times the magnitude of the difference $\boldsymbol{N} - \boldsymbol{G}_n$. Now,

$$\boldsymbol{N} - \boldsymbol{G}_n = \frac{-m(\Delta + g)}{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}\boldsymbol{n}, \tag{10}$$

and so the magnitude of this vector is

$$\frac{m(\Delta + g)}{\left(\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1\right)^{\frac{1}{2}}}. \tag{11}$$

The frictional force is this magnitude times the coefficient of friction μ times the unit vector $-\frac{\boldsymbol{r}'}{|\boldsymbol{r}'|}$. We therefore find that the frictional force is

$$\boldsymbol{f} = -\frac{\mu m(\Delta + g)}{\sqrt{\left(\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1\right)\left((x')^2 + (y')^2 + (z')^2\right)}}\boldsymbol{r}'. \tag{12}$$

Using the equation $z = f(x, y)$, we can write $z' = \frac{\partial f}{\partial x}x' + \frac{\partial f}{\partial y}y'$, and hence get

$$\boldsymbol{f} = \frac{-\mu m(\Delta + g)}{\sqrt{\left(\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1\right)\left(\left(1 + \left(\frac{\partial f}{\partial x}\right)^2\right)(x')^2 + 2\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}x'y' + \left(1 + \left(\frac{\partial f}{\partial y}\right)^2\right)(y')^2\right)}}\boldsymbol{r}'. \tag{13}$$

Thus the tangential component of the net force is

$$\begin{aligned}\boldsymbol{F}_t = \boldsymbol{G}_t + \boldsymbol{f} = {} & \frac{-mg}{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}\left(\frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} + \left(\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right)\boldsymbol{k}\right) + \\ & + \frac{-\mu m(\Delta + g)}{\sqrt{\left(\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1\right)\left(\left(1 + \left(\frac{\partial f}{\partial x}\right)^2\right)(x')^2 + 2\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}x'y' + \left(1 + \left(\frac{\partial f}{\partial y}\right)^2\right)(y')^2\right)}}\left(x'\boldsymbol{i} + y'\boldsymbol{j} + z'\boldsymbol{k}\right).\end{aligned} \tag{14}$$

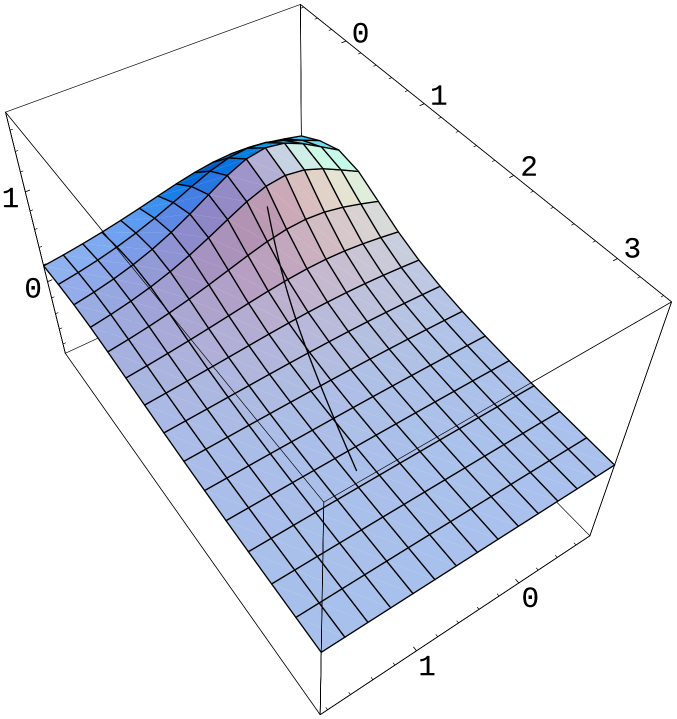
The last term in (14) becomes indeterminate at points where $\boldsymbol{r}' = \boldsymbol{0}$. Thus, in order to get a determinate system of differential equations, we will have to forego the seemingly natural assumption that the particle begins to slide from rest. Some initial impulse imparts a velocity to it, which we will assume is in the direction of maximum decrease of z. That is, we shall assume that $x'(0) = -\rho\frac{\partial f}{\partial x}$ and $y'(0) = -\rho\frac{\partial f}{\partial y}$, where $\rho > 0$.

As already stated, the normal component of the force is

$$\boldsymbol{F}_n = \frac{-m\Delta}{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}\left(\frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} - \boldsymbol{k}\right). \tag{15}$$

The total force on the particle is then

$$m\boldsymbol{r}''(t) = \boldsymbol{F}_t + \boldsymbol{F}_n. \tag{16}$$



Mathematica-produced trajectory for migration of a particle. The surface is assumed to have the equation $z = \frac{1}{x^2+y^2+1}$.

Component by component, then, the differential equations that have to be satisfied are

$$
\begin{aligned}
x'' &= \frac{-(\Delta+g)}{\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2+1}\frac{\partial f}{\partial x}+\\
&\quad+\frac{-\mu(\Delta+g)x'}{\sqrt{\left(\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2+1\right)\left(\left(1+\left(\frac{\partial f}{\partial x}\right)^2\right)(x')^2+2\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}x'y'+\left(1+\left(\frac{\partial f}{\partial y}\right)^2\right)(y')^2\right)}},\\
y'' &= \frac{-(\Delta+g)}{\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2+1}\frac{\partial f}{\partial y}+\\
&\quad+\frac{-\mu(\Delta+g)y'}{\sqrt{\left(\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2+1\right)\left(\left(1+\left(\frac{\partial f}{\partial x}\right)^2\right)(x')^2+2\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}x'y'+\left(1+\left(\frac{\partial f}{\partial y}\right)^2\right)(y')^2\right)}},\\
z'' &= \frac{-g}{\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2+1}\left(\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2\right)-\frac{\mu(\Delta+g)z'}{\sqrt{\left(\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2+1\right)\left((x')^2+(y')^2+(z')^2\right)}}.
\end{aligned}
\tag{17}
$$

The equation for z looks formidable, but fortunately we do not need it. We can determine z from the equation $z = f(x, y)$ once we know x and y. Thus we need only the first two equations of (17). Assuming that the particle begins to slide from rest at time $t = 0$, starting from the position (x_0, y_0, z_0), where, of course, $z_0 = f(x_0, y_0)$, we then have the following initial-value problem:

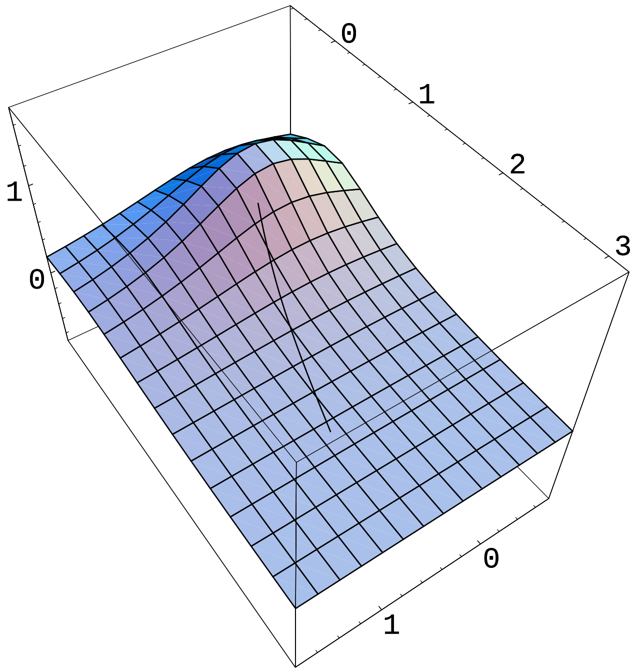
$$\begin{aligned}
x'' = {} & \frac{-(\Delta + g)}{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1} \frac{\partial f}{\partial x} + \\
& + \frac{-\mu(\Delta + g)x'}{\sqrt{\Big(\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1\Big)\Big(\big(1 + \left(\frac{\partial f}{\partial x}\right)^2\big)(x')^2 + 2\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}x'y' + \big(1 + \left(\frac{\partial f}{\partial y}\right)^2\big)(y')^2\Big)}}, \\
y'' = {} & \frac{-(\Delta + g)}{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1} \frac{\partial f}{\partial y} + \\
& + \frac{-\mu(\Delta + g)y'}{\sqrt{\Big(\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1\Big)\Big(\big(1 + \left(\frac{\partial f}{\partial x}\right)^2\big)(x')^2 + 2\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}x'y' + \big(1 + \left(\frac{\partial f}{\partial y}\right)^2\big)(y')^2\Big)}}, \\
& x(0) = x_0, \quad x'(0) = -\rho\frac{\partial f}{\partial x}, \\
& y(0) = y_0, \quad y'(0) = -\rho\frac{\partial f}{\partial y}.
\end{aligned} \tag{18}$$

The equations (18) can be solved for $x(t)$ and $y(t)$, after which $z(t)$ can be found as $f\big(x(t), y(t)\big)$. However, for even moderately complicated functions $f(x, y)$, these equations are far too messy to solve in closed form, and numerical methods must be used. The *Mathematica* notebook `geology.nb` contains instructions for solving them and sample output.

Small velocities. Equations (18) describe the motion of a particle at all speeds. However, for particles moving over a landscape, the velocity is so small that the function Δ, which is a quadratic function of the velocity, can be neglected. At small velocities, the particle is nearly at rest, in which case the frictional force is relatively large and can be assumed to act oppositely to the net tangential gravitational force $\boldsymbol{G}_t$, which is in the direction of steepest decrease of z. In this case, we can assume that the particle begins to slide from a resting position. The result is the following simpler set of equations for the motion:

$$\begin{aligned}
x'' &= \frac{-g}{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}\left(1 - \frac{\mu}{\sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}}\right)\frac{\partial f}{\partial x}, \\
y'' &= \frac{-g}{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}\left(1 - \frac{\mu}{\sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}}\right)\frac{\partial f}{\partial y}, \\
& x(0) = x_0, \quad x'(0) = 0, \\
& y(0) = y_0, \quad y'(0) = 0.
\end{aligned} \tag{19}$$

After $x(t)$ and $y(t)$ are determined, $z(t)$ is found again as $f\big(x(t), y(t)\big)$. The acceleration described by Eqs. (19) is always in the direction of steepest descent. The factor $\left(1 - \frac{\mu}{\sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}}\right)$ must be positive. Otherwise the frictional force works in the same direction as gravity, which is possible for a rapidly moving particle that slides past a local minimum of z, but not in this low-velocity model. That requirement, of course, merely says that the coefficient of friction must be less than the reciprocal of the slope of the gradient. Again, that is in accordance with intuition, since the magnitude of the frictional force is given by $\mu|\boldsymbol{G}_n| = \mu|\boldsymbol{G}_t|/\sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial x}\right)^2}$. In order to get motion, this magnitude must be less than $|\boldsymbol{G}_t|$.



Mathematica-produced trajectory for slow migration