A SUNDIAL PROBLEM

The problem. All sundials admit a certain error due to incorrect alignments. Since true solar time can be calculated from standard time knowing the observer's terrestrial longitude and the relation between true solar time and mean solar time that is expressed in the periodic relation known (misleadingly) as the equation of time, it is possible in principle to determine the nature of the misalignment of a sundial by comparing readings of time taken from the sundial with the true times. The following pages present one possible implementation of such a program.

The assumptions. We ignore any imperfections in the machining of the sundial itself. We assume it consists of a perfectly flat planar face ruled into quarters by two perfectly perpendicular NS- and WE-axes. At the intersection of the axes a gnomon of length l is implanted making an angle with the NS-axis equal to the terrestrial latitude of the location of the sundial (so that, if the sundial is perfectly positioned, the gnomon will point to the north celestial pole). We shall take a set of three mutually perpendicular axes fixed in the sundial, with the x axis (the NS-axis) pointing south, the y-axis (the WE-axis) pointing east, and the z-axis pointing "upward" from the plane of the sundial face. Since all corrections will be described in terms of rotations about the y and z axes, we make the convention that a rotation about either of these axes is positive if it is counterclockwise when viewed by an observer looking in the direction the axis points. Finally we make the usual assumption that the sundial is between the Arctic Circle and the Tropic of Cancer, at latitude L degrees ($23.5° < L < 66.5°$).

We also need two sets of spherical latitude and longitude coordinates. The first is the celestial sphere itself, in which latitude 90° is the north celestial pole. Since longitude can be measured from any point on the equator, we assume that the prime meridian passes through the local zenith. Thus longitude zero is true south. Longitude east is negative and longitude west is positive (so that the sun's longitude increases during each day). Celestial longitude will be denoted θ and latitude φ.

The second set of spherical coordinates is centered at the base of the gnomon of the sundial. Its equator is the plane of the sundial face, and its prime meridian is the meridian through the gnomon and the NS-axis. Longitude and latitude in sundial coordinates will be denoted $\hat{\theta}$ and $\hat{\varphi}$. If the sundial is perfectly positioned, the two prime meridians will coincide and the pole of the sundial system will have celestial coordinates $\theta = 0$ and $\varphi = L$. The sundial face will then be a horizontal plane relative to the earth's surface.

Precise statement of the problem. Assume that, through minor errors of adjustment, the pole of the sundial system of spherical coordinates has celestial coordinates (θ_0, φ_0), and that its prime meridian makes an acute angle α (which may be positive or negative) with the celestial meridian $\theta = \theta_0$. If we can discover the values of the three angles $\theta_0, \varphi_0, \alpha$ from comparisons of sundial time with true local solar time, the correction can easily be made by carrying out the following four rotations on the sundial *in the stated order*:

$i)$ Rotate about the z-axis through angle $-\alpha$. This rotation causes the sundial prime meridian to coincide with the meridian $\theta = \theta_0$.

$ii)$ Rotate about the y-axis through angle $90° - \varphi_0$. This rotation brings the sundial pole into coincidence with the north celestial pole.

$iii)$ Rotate about the z-axis through angle $-\theta_0$. This rotation brings the entire sundial spherical coordinate system into coincidence with the celestial spherical coordinate system.

$iv)$ Rotate about the y-axis through angle $L - 90°$. This rotation leaves the two prime meridians concident and puts the pole of the sundial system at the desired location $\theta = 0, \varphi = L$.

Thus the entire problem is now reduced to finding the three angles $\theta_0, \varphi_0, \alpha$. In order to do so, we need to make two connections. The first connection is the relation between the current sundial reading, which we give as the coordinates $\big(x(t), y(t)\big)$ of the shadow of the tip of the gnomon, and the sun's current sundial longitude and latitude $\hat{\theta}(t)$ and $\hat{\varphi}(t)$. The connections are as follows:

$$\begin{aligned} x &= -l\cos L - \frac{l\sin L\cos\hat{\theta}}{\tan\hat{\varphi}}, \\ y &= \frac{l\sin L\sin\hat{\theta}}{\tan\hat{\varphi}}, \end{aligned} \tag{1}$$

and the inverse relations

$$\begin{aligned} \hat{\theta} &= -\arctan\left(\frac{y}{x + l\cos L}\right), \\ \hat{\varphi} &= \arctan\left(\frac{l\sin L}{\sqrt{y^2 + (x + l\cos L)^2}}\right). \end{aligned} \tag{2}$$

Since x and y can be read at any time and l and L are known, we can compute the sundial spherical coordinates of the sun at any time using Eqs. (2).

The second connection we need is the relation between the celestial and sundial spherical coordinates of a given point. These are given by the spherical law of sines and the spherical law of cosines:

$$\begin{aligned} \sin(\alpha + \hat{\theta})\cos\hat{\varphi} &= \sin(\theta - \theta_0)\cos\varphi, \\ \sin\varphi &= \sin\hat{\varphi}\sin\varphi_0 - \cos\hat{\varphi}\cos\varphi_0\cos(\hat{\theta} + \alpha), \\ \sin\hat{\varphi} &= \sin\varphi\sin\varphi_0 + \cos\varphi\cos\varphi_0\cos(\theta - \theta_0). \end{aligned} \tag{3}$$

We hope to use known values of θ, φ, $\hat{\theta}$, and $\hat{\varphi}$ to solve these equations for θ_0, φ_0, and α.

We shall assume that on a given day, at a given time t (measured in hours before and after noon, $-12 < t < 12$), the sun has celestial spherical coordinates $\theta = 15t$, $\varphi = d$, where d is the declination of the sun. For simplicity we shall assume that d is constant for a given day. (Over the daylight period when the sundial works, d can vary by at most 0.1° above and below its average value for the day; this maximum occurs around the equinoxes. The variation of d over one day near the solstices is extremely small.) Hence if we simply know the date (which gives the declination) and the true solar time t, which can be worked out from standard time, terrestrial latitude, and the equation of time, we can get the current values of θ and φ to put into these equations. As mentioned above, the values of $\hat{\theta}$ and $\hat{\varphi}$ come from reading the sundial.

The solution of the equations. We are now in a position to solve these equations. We shall assume that the values of θ, φ, $\hat{\theta}$, and $\hat{\varphi}$ are known, and solve for θ_0, φ_0, and α. The trick is to choose two observation times so that the solution is as simple as possible. We shall choose "sundial noon" as the first observation. At that time we observe the sundial to find the shadow of the gnomon at a point $(x_1, 0)$, so that $\hat{\theta}_1 = 0$ and $\hat{\varphi}_1 = -\arctan\left(\frac{l\sin L}{x_1 + l\cos L}\right)$. Note that $x_1 + l\cos L$ is a negative number. Knowing that the true solar time is t_1, we work out the values $\theta_1 = 15t_1$, $\varphi_1 = d$. These values allow us to express θ_0 from the first of Eqs. (3) in terms of α and known quantities:

$$\theta_0 = \theta_1 - \arcsin\left(\frac{\cos\hat{\varphi}_1\sin\alpha}{\cos d}\right). \tag{4}$$

We can express φ_0 in terms of α and known quantities by using the law of cosines. First we note that the angle $\psi = \arctan(\cos\alpha\cot\hat{\varphi}_1)$ has the property that $\cos\psi = \frac{\sin\hat{\varphi}_1}{\sqrt{\sin^2\hat{\varphi}_1 + \cos^2\hat{\varphi}_1\cos^2\alpha}}$ and $\sin\psi = \frac{\cos\hat{\varphi}_1\cos\alpha}{\sqrt{\sin^2\hat{\varphi}_1 + \cos^2\hat{\varphi}_1\cos^2\alpha}}$. (The angle ψ is known once α is found.) Now the second equation of Eqs. (3)

says $\sin d = \sin\hat{\varphi}_1 \sin\varphi_0 - \cos\hat{\varphi}_1 \cos\varphi_0 \cos\alpha$, which says $\sin d = \sqrt{\sin^2\hat{\varphi}_1 + \cos^2\hat{\varphi}_1\cos^2\alpha}(\sin\varphi_0\cos\psi - \cos\varphi_0\sin\psi) = \sqrt{1-\cos^2\hat{\varphi}_1\sin^2\alpha}\,\sin(\varphi_0-\psi)$. Therefore

$$\varphi_0 = \arcsin\Big(\frac{\sin d}{\sqrt{1-\cos^2\hat{\varphi}_1\sin^2\alpha}}\Big) + \arctan(\cos\alpha\cot\hat{\varphi}_1). \tag{5}$$

The final problem is to solve for α. We need a different set of observations to do this. We warn in advance that the procedure is rather arduous, and will require some new symbols in order to consolidate some very complicated expressions.

Our second set of observations is made at true local solar noon, when $\theta = 0$, $\varphi = d$, $\hat{\theta} = \hat{\theta}_2$ and $\hat{\varphi} = \hat{\varphi}_2$, the latter values being computed using Eqs. (2) from the coordinates $\big(x_2(t), y_2(t)\big)$ read from the sundial at true solar noon. All four of these values are known. We now insert them into the first of Eqs. (3), getting

$$\sin(\alpha + \hat{\theta}_2)\cos\hat{\varphi}_2 = -\sin\theta_0\cos d.$$

Using the expression (4) for θ_0 and the addition formula for sine on both sides, recalling that $\cos(\arcsin z) = \sqrt{1-z^2}$, we get the equation

$$(\cos\hat{\theta}_2\cos\hat{\varphi}_2 - \cos\theta_1\cos\hat{\varphi}_1)\sin\alpha + \sin\hat{\theta}_2\cos\hat{\varphi}_2\cos\alpha = -\sin\theta_1\sqrt{\cos^2 d - \cos^2\hat{\varphi}_1\sin^2\alpha}.$$

We square both sides of this equation and transpose all terms involving α to the left-hand side, getting

$$A\sin^2\alpha + B\sin\alpha\cos\alpha + C\cos^2\alpha = \sin^2\theta_1\cos^2 d, \tag{6}$$

where

$$\begin{aligned} A &= (\cos\hat{\theta}_2\cos\hat{\varphi}_2 - \cos\theta_1\cos\hat{\varphi}_1)^2 + \sin^2\theta_1\cos^2\hat{\varphi}_1, \\ B &= 2\sin\hat{\theta}_2\cos\hat{\varphi}_2(\cos\hat{\theta}_2\cos\hat{\varphi}_2 - \cos\theta_1\cos\hat{\varphi}_1), \\ C &= \sin^2\hat{\theta}_2\cos^2\hat{\varphi}_2. \end{aligned} \tag{7}$$

The numerical values of A, B, and C are all computable from known data. We now use double-angle formulas to write Eq. (6) as

$$A\Big(\frac{1}{2} - \frac{1}{2}\cos 2\alpha\Big) + \frac{1}{2}B\sin 2\alpha + C\Big(\frac{1}{2} + \frac{1}{2}\cos 2\alpha\Big) = \sin^2\theta_1\cos^2 d,$$

that is,

$$B\sin 2\alpha + (C-A)\cos 2\alpha = 2\sin^2\theta_1\cos^2 d - (A+C). \tag{8}$$

We now choose the angle ζ so that

$$\frac{B}{\sqrt{B^2+(A-C)^2}} = \cos\zeta, \quad \frac{A-C}{\sqrt{B^2+(A-C)^2}} = \sin\zeta.$$

(It is important to use these equations so as to know what quadrant the angle ζ is in. There are only two possible choices for ζ, namely $\zeta = \arctan\Big(\frac{A-C}{B}\Big)$ or $\zeta = 180^\circ + \arctan\Big(\frac{A-C}{B}\Big)$.) We then, finally, have

$$\sin(2\alpha - \zeta) = \frac{2\sin^2\theta_1\cos^2 d - (A+C)}{\sqrt{B^2+(A-C)^2}},$$

so that α is given by one of the two following equations

$$\alpha = \frac{1}{2}\Bigg(\arctan\Big(\frac{A-C}{B}\Big) + \arcsin\Big(\frac{2\sin^2\theta_1\cos^2 d - (A+C)}{\sqrt{B^2+(A-C)^2}}\Big)\Bigg), \tag{9}$$

$$\alpha = 90^\circ + \frac{1}{2}\Bigg(\arctan\Big(\frac{A-C}{B}\Big) + \arcsin\Big(\frac{2\sin^2\theta_1\cos^2 d - (A+C)}{\sqrt{B^2+(A-C)^2}}\Big)\Bigg). \tag{9$'$}$$