Kovalevskaya's Mathematical Achievements

By Roger Cooke

The present essay is a brief analysis of the mathematical achievements of Sof'ya Vasil'evna Kovalevskaya (1850–1891). For a sketch of her life, see the companion essay posted at this website. It is of course necessary to place her achievements in the context of the mathematics that came before and after her time. For that reason this essay is divided into five parts. In the first part we go back to the seventeenth and eighteenth centuries to describe the mathematics that she inherited from her predecessors. In the second part we describe her education and the mathematical areas in which she specialized. In parts three and four we analyze her two greatest results, and in part five we look briefly at her minor papers and give an overall assessment of her place in the history of mathematics. Although this essay makes some mathematical demands on the reader, at least to the extent of assuming that partial differential equations are a familiar concept, I have tried to keep the descriptions as elementary as possible.

From Calculus to Modern Algebra and Analysis. Although the problems that motivated them come from geometry and physics, the basic techniques of calculus—differentiation, integration, and infinite series—are essentially algebraic subjects. In fact the equation "algebra + infinitesimals = calculus" is a fair approximation to the truth. Infinitesimal methods originally arose in geometry in problems of constructing tangents to curves or finding the lengths of curved lines and the areas enclosed by such lines, and the areas of curved surfaces and the volumes enclosed by such surfaces. These methods could not be systematized, however, until they were suitably formulated using the symbolic methods of algebra. (Algebra, the study of equations, was long carried out in the ordinary language of speech. It was not until the late sixteenth century that the shorthand symbolism we now associate with algebra began to be used.) Once this labor had been performed, a powerful mathematical technique arose through the application of the fundamental theorem of calculus to unite problems that originally seemed to have no connection. The laws of nature turned out to be differential equations. Differential calculus became important as the way of deriving such equations, while integral calculus and infinite series were the way to solve them. In this way many deep problems of mechanics, heat, and ultimately electricity, magnetism, and light, all became amenable to study through models that involved differential equations.

Algebra, differentiation, and integration. The full understanding of the nature of these algebraic problems, however, required a great deal of basic research into the nature of numbers and the fundamental problem of algebra: finding the roots of a polynomial. It required, for example, complex numbers to reveal hitherto hidden connections between the exponential and trigonometric functions in the formula $e^{ix} = \cos(x) + i\sin(x)$ and to explain why the formula $\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots$ is valid only when $-1 < x < 1$. A very important discovery was that the operation of integration leads out of the domain of algebraic functions, while differentiation does not. For example, the integral of the function $f(t) = \frac{1}{t}$ is $\ln(t)$. The effort to explain these mysteries led mathematicians far afield and generated a wide variety of techniques that seemingly had little to do with one another, despite having a common source in the problems of solving differential equations.

Obstacles to the solution of differential equations. The solution of a differential equation could be stymied in two different ways. First, it might not be possible to manipulate the equation in such a way that the solution could be exhibited as an integral. Second, when the solution was exhibited in this way (reduced to quadratures, as the mathematicians of the time said), the integrals might not be expressible in terms of known functions. An example of the first phenomenon occurs in the case of Bessel's equation

$$x^2y''(x) + xy'(x) + (p^2 - x^2)y(x) = 0,$$

which arises when the variables are separated in Laplace's equation in cylindrical coordinates. The second phenomenon occurs even more frequently, for example in the equation of pendulum motion

$$y''(t) + \omega^2 \sin\big(y(t)\big) = 0,$$

which can be reduced to quadratures as

$$dt = \frac{dy}{\sqrt{C - 2\omega^2 \cos y}},$$

or, with the substitution $u = \cos y$,

$$dt = -\frac{du}{\sqrt{(C - 2\omega^2 u)(1 - u^2)}}.$$

These last integrals turn out to be inexpressible in terms of the elementary algebraic, trigonometric, and exponential functions.

Power-series solutions. In the first case the pioneers of calculus did not hesitate to use infinite series, regarding a function as known if its Taylor coefficients could be generated recursively; later they added trigonometric (Fourier) series to their repertoire of techniques. In the second case an integral that could not be evaluated was given a name, thereby becoming a new function whose relations to known functions could be investigated. Thus, for example, integrals involving the square root of a cubic or quartic polynomial were classified and generically referred to as *elliptic functions* because of their connection with the rectification of the ellipse.

Integrals of algebraic functions. The systematic study of algebraic integrals proceeded in parallel with the development of algebra itself. Both subjects were advanced spectacularly in the work of the short-lived Norwegian prodigy Niels Henrik Abel (1802–1829). In his youth Abel had the ambition of settling these problems once and for all by finding a general formula for the roots of any polynomial and a general form for the integral of any algebraic function. Eventually he realized that such formulas could not be obtained and produced a proof that the general quintic equation could not be solved by a finite number of algebraic operations. As for integrals, he brought a great deal of order into this area and produced what is arguably the most profound theorem of the early nineteenth century, the theorem now known by his name. He considered two variables x and y constrained to satisfy a polynomial equation $p(x, y) = 0$. A general algebraic integral, now known as an *abelian integral*, is an integral of the form

$$\int R(x, y)\, dx,$$

where $R(x, y)$ is a rational function of the two variables. For example, the elliptic integral

$$\int \frac{1}{\sqrt{1 - x^4}}\, dx$$

is of this form with $p(x, y) = x^4 + y^2 - 1$ and $R(x, y) = \frac{1}{y}$.

Abel's great result was that for any given polynomial $p(x, y)$ and rational function $R(x, y)$ there is a number g such that the sum of any given number of indefinite integrals can be expressed as a sum of g integrals whose limits of integration are algebraic functions of the limits in the given integrals.

Jacobi and theta functions. Abel died without being able to explore fully the ramifications of this theorem, but his work was ably continued by C.G.J. Jacobi (1804–1851). (In fact it was Jacobi who proposed the name *Abel's theorem*.) In a series of papers devoted to elliptic functions and extending to more general algebraic integrals, Jacobi showed that these algebraic integrals could not be understood fully without the use of complex variables, and that their inverse functions were always functions of n complex variables having $2n$ independent periods. In the case of elliptic functions, for which Abel's number g equals 1, Jacobi showed that the inverse functions could be elegantly expressed as quotients of functions that later came to be known as *theta functions*. The two theta functions that he introduced were

$$\Theta\Big(\frac{2Kx}{\pi}\Big) = 1 - 2q\cos 2x + 2q^4\cos 4x - 2q^9\cos 6x + \cdots$$
$$H\Big(\frac{2Kx}{\pi}\Big) = 2\sqrt[4]{q}\sin x - 2\sqrt[4]{q^9}\sin 3x + 2\sqrt[4]{q^{25}}\sin 5x - 2\sqrt[4]{q^{49}}\sin 7x + \cdots .$$

The constant K here is known as the complete elliptic integral of first kind. It depends on a parameter k via the formula

$$K = \int_0^1 \frac{1}{\sqrt{(1-t^2)(1-k^2t^2)}}\,dt.$$

In 1849 Jacobi was able to use these theta functions to give a very elegant expression of the functions that describe the motion of a rigid body free of external torque, a case of motion that had been studied previously by Leonhard Euler (1707–1783).

Mathematical complications in the two techniques. The two techniques—reduction to quadratures and power-series expansions—turned out to share a common domain, namely the complex numbers. We noted above that algebraic functions really cannot be understood on any other basis. Now the theory of functions of a complex variable was constructed by Kovalevskaya's adviser Karl Weierstrass (1815–1897) strictly on the basis of power series. Weierstrass had been anticipated in this work by Augustin-Louis Cauchy (1789–1856), who had based this theory on the use of complex integrals, but had also shown that the functions involved could always be represented by power series. Weierstrass' contemporary Georg Friedrich Riemann (1826–1866) based his development of the subject on geometric transformations, but he too employed power series. The theory of analytic functions of a complex variable was the vantage point from which Weierstrass and his students studied both algebraic functions and differential equations. One peculiar consequence of this approach was that the fundamental physical variable, time, which is intuitively a linearly ordered set, must be represented as a complex variable, which is *not* linearly ordered in any natural way.

General algebraic functions. Jacobi was the first to note that if $\Pi(x)$ is an elliptic integral, Abel's theorem can be stated as the equation

$$\Pi(x) + \Pi(y) = \Pi(a),$$

where a is an algebraic function of x and y. At the same time he noted that for a more complicated integral, say where

$$\Pi(x) = \int_0^x \frac{1}{\sqrt{1-t-t^5}}\,dt,$$

Abel's theorem seems to lead to indeterminacy, since it provides only one equation

$$\Pi(x) + \Pi(y) + \Pi(z) = \Pi(a) + \Pi(b),$$

in which a and b are two algebraic functions of x, y, and z. In 1832 he proposed that a second integral be considered, namely

$$\Phi(x) = \int_0^x \frac{t}{\sqrt{1-t-t^5}}\,dt,$$

and posed the problem of finding x and y in terms of u and v from the equations

$$\Pi(x) + \Pi(y) = u, \quad \Phi(x) + \Phi(y) = v.$$

This problem, in general form, became known as the *Jacobi inversion problem*. A complete solution of it required 25 years of work, the final step being taken almost simultaneously in 1856 by Riemann and Weierstrass, using two very different approaches.

Weierstrass' development of algebraic function theory. Weierstrass was so intrigued by the fact that Riemann and he had both solved the same problem from different perspectives that he withdrew his paper in order to investigate the relation between them. By the mid-1860s he had organized the subject to his own satisfaction, in a large set of notes published as Vol. 4 of his collected works. The secret of solving the Jacobi inversion problem turned out to be multivariable theta functions, whose theory was developed during the 1840s and 1850s.

Bearing in mind that elliptic integrals were originally encountered in solving simple problems of mechanics, such as that of pendulum motion, one would naturally inquire about the practical application, if any, of the more complicated algebraic integrals or the theta functions of two variables used to express them.

Power series solutions. The power series technique also required mathematical investigation. The simple problem was the following: A differential equation can be used to generate the coefficients of a power series recursively. To take, for example, the Bessel equation of order 0

$$x^2y''(x) + xy'(x) - x^2y(x) = 0$$

with the initial conditions $y(0) = 1$, $y'(0) = 0$, and assuming a power-series expansion

$$y(x) = \sum_{n=0}^{\infty} a_n x^n,$$

one finds the relations $a_0 = 1$, $a_1 = 0$ and

$$n(n-1)a_n + na_n + a_{n-2} = 0,$$

that is,

$$a_n = -\frac{a_{n-2}}{n^2},$$

from which it is easy to prove recursively that

$$a_{2n-1} = 0, \quad a_{2n} = (-1)^n \frac{1}{2^{2n}(n!)^2},$$

and hence that the solution of this equation, usually denoted $J_0(x)$, is

$$J_0(x) = \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{x}{2}\right)^{2n}}{(n!)^2}.$$

It is not difficult to verify that in this particular case the series converges for all values of x (real and complex), and therefore provides a genuine solution of the initial-value problem. The difficulty mathematicians see in the technique, however, is that it tells only what the coefficients of a series *must* be if the series is to represent a function satisfying the differential equation. It is necessary at the very least to verify that the recursively generated series does converge. It would be desirable to prove a general theorem to the effect that a given differential equation has an analytic solution. For Weierstrass this was particularly important, since his definition of an analytic function involved a convergent power series. If such a theorem could be proved, then a differential equation itself could be used as the definition of an analytic function. Such was the theme of a paper Weierstrass wrote in 1841 (but did not publish until 1894).

These two topics, abelian integrals and power-series solutions of differential equations, were the areas in which Kovalevskaya most distinguished herself. As we have seen, they have a common root in the basic problem of solving differential equations and a common domain in the analytic functions of a complex variable. A third common element is that both areas were greatly advanced by Jacobi, whose influence on both Kovalevskaya and Weierstrass is especially noticeable in the analytic style in which all three wrote.

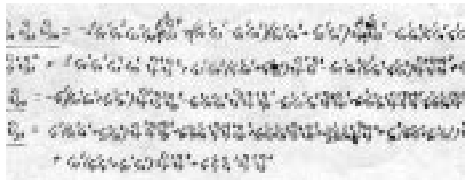
Kovalevskaya's Education. Kovalevskaya studied with Weierstrass for four years, from 1870 to 1874. Although she was not allowed to attend his lectures, he received her in his home on Sundays and gave her lessons there. Some idea of the contents of this private instruction can be gained from the list of courses he was giving at the University of Berlin at this time and his letters to her. The courses listed in his collected works are the following:

Winter 1870–1871	Elliptic Functions
Summer 1871	Recent Synthetic Geometry
	Selected Problems of Geometry and Mechanics Solvable by Elliptic Functions
Winter 1871–1872	Abelian Functions
Summer 1872	Introduction to Analytic Functions
	Calculus of Variations
Winter 1872–1873	Elliptic Functions
Summer 1873	Elements of Recent Synthetic Geometry
	Selected Problems of Geometry and Mechanics Solvable by Elliptic Functions
Winter 1873–1874	Abelian Functions

It seems clear from the correspondence that Weierstrass taught Kovalevskaya essentially the same things he taught his other students. He was teaching abelian functions in January 1872, and on 4 January he sent her a letter cancelling their appointment for the next day and enclosing some notes on the topic of their next appointment. These notes are concerned with hyperelliptic functions as a special case of abelian functions, and Weierstrass says explicitly that he needs these notes for his next lecture. Letters from November 1872 contain discussions of problems in the calculus of variations, consistent with the offering of that course in the summer semester. However, several other letters from the same time discuss theta functions. A year later, when Weierstrass was teaching geometry and its applications, the letters discuss minimal surfaces.

Kovalevskaya's dissertations. Weierstrass mentioned three papers of Kovalevskaya when he wrote to the University of Göttingen to request that she be granted the doctoral degree. By far the most important of these was her paper on analytic solutions of partial differential equations, which contains the result now known as the Cauchy–Kovalevskaya theorem. Almost any university, however, would have awarded the degree on the basis of a second paper that she wrote, on abelian integrals that degenerate to elementary functions. For good measure Weierstrass included a third paper, an analysis of the shape of the cross-section of a ring around a planet, carried out on the assumption that a thin layer of liquid on the surface of the ring would be in equilibrium. The only paper she published at the time was the first of these three. Judging from the correspondence, one would conclude that this paper must have been written in a very short time. The dissertation was completed in the summer of 1874, yet no letter mentions this topic until 6 May of that year. The topic then occurs frequently in letters through 4 July, on which date Weierstrass sent her a long letter containing some formulas that actually occur in her dissertation. His letter to Lazarus Fuchs requesting the degree was written on almost the same day.

Kovalevskaya's specialty. Even 125 years ago the subject of algebraic functions was sufficiently complicated that a thorough mastery of it required several years of study. By the time she had finished studying with Weierstrass Kovalevskaya was one of the world's experts in this area. In addition, she had a thorough grounding in the more analytic parts of geometry and differential equations. These were the subjects she was prepared to teach when, eight years later, she took up her position at Stockholm University.



A small portion of a page almost solidly covered by theta functions, written by Kovalevskaya. Courtesy Institut Mittag-Leffler.

Partial Differential Equations. As mentioned above, the first person to ask whether a differential equation automatically generates a convergent power-series solution via the so-called method of undetermined coefficients. Cauchy first considered this problem in his lectures at the École Polytechnique in 1823, then again in a series of memoirs in 1835. In 1842 he took a third approach to the problem and established the existence of an analytic solution to a partial differential equation having analytic coefficients using a method that he called the *calcul des limites*, but which is commonly called the *method of majorants* in English. The idea behind the method is that, when a power series $\sum a_n x^n$ converges for any non-zero value of x, say $x = x_0$, it must converge for all values of x such that $|x| < |x_0|$ by comparison with a geometric series. That is, if

$$\sum a_n x_0^n$$

converges, then its terms must tend to zero. As a consequence there is a constant M such that $|a_n x_0^n| \le M$ for all n. But then $|a_n x^n| \le M\left|\frac{x}{x_0}\right|^n$, so that for $|x| < |x_0|$, the terms of the original series are majorized by those of the geometric series

$$\sum M\Bigl(\frac{x}{x_0}\Bigr)^n = \frac{M}{1 - \frac{x}{x_0}}.$$

This observation, that the Cauchy kernel $\frac{1}{1-z}$ has a Taylor series with positive coefficients that majorizes the terms of any convergent power series, provided Cauchy, and almost simultaneously Weierstrass, with a technique for proving that the method of majorants must produce a convergent series. The idea is to exploit the positivity of the coefficients of this series, replace all the analytic data in the equation with the Cauchy kernel, and thereby generate a power series that majorizes, term by term, the series generated by the original equation. Since the equation with the Cauchy kernel as data can be solved explicitly and shown to have an analytic solution, it follows that this series converges and hence that the original equation has an analytic solution.

Except for technical details (which are considerable) the process just described is the essence of the Cauchy–Kovalevskaya theorem.

Cauchy's 1842 papers. Cauchy considered a quasi-linear partial differential equation

$$D_t \varpi = A D_x \varpi + B D_y \varpi + \cdots + K,$$

where the coefficients $A, B, \ldots, K$ are functions of both the unknown ϖ and the independent variables t, x, $y, \ldots$. For the case when the coefficients are $A = \frac{a}{xyz\cdots t}$, $B = \frac{b}{xyz\cdots t}$, $\ldots$, $K = \frac{k}{xyz\cdots t}$, Cauchy solved this differential equation with the initial conditions

$$\varpi = \omega, \quad D_x \varpi = 0, \quad D_y \varpi = 0, \ldots,$$

at the point $t = \tau$, giving the solution as the integral

$$\log\Bigl(\frac{t}{\tau}\Bigr) = \int_0^{\varpi - \omega} (\varpi - \theta)\Bigl(x + a\frac{\theta}{k}\Bigr)\Bigl(y + b\frac{\theta}{k}\Bigr)\cdots \frac{1}{k}\, d\theta.$$

Since, as remarked above, the terms of the Taylor series of this analytic function majorize those that result from any given analytic coefficients $A, \ldots, K$, it follows that the initial-value (Cauchy) problem for this quasi-linear equation must have an analytic solution.

The method of majorants was designed to put a foundation under another method, namely the method of undetermined coefficients. It gives a sufficient condition for the method of undetermined coefficients to "work." However, when the method of majorants applies, it yields only a sufficient condition for convergence, not a necessary one. Given a particular equation, one could apply the method of majorants and thereby justify the method of undetermined coefficients. However, the only way of knowing if the method would work was to try it. Some higher-level theorem would be desirable, a theorem that would give a characteristic of a differential equation by means of which one could determine whether the method of undetermined coefficients would work. The method of majorants should appear only in the proof of the theorem, not in its application.

Jacobi's canonical and normal form. A property of differential equations that makes their solution more feasible was discovered by Jacobi,who defined an equation to be in "normal form" if it could be solved for the highest order derivative with respect to some variable (and apparently highest order meant the pure derivative of order n with respect to some variable when the equation is of order n).* He did not make any connection between normal form and the method of undetermined coefficients. That connection was an important part of Kovalevskaya's achievement.

He called the same concept "canonical form" in a later paper, published in Vol. 64 of the *Journal*, saying that "a system of ordinary differential equations is *non-canonical* if the equations contain the highest-order derivatives of the dependent variables in such a way that they cannot be determined from the equations." (That is, the equations cannot be solved for the highest-order derivatives.) In the papers just mentioned he gave a combinatorial procedure for reducing equations to canonical or normal form.

Kovalevskaya's theorems. Kovalevskaya considered a very general ordinary differential equation

$$G\Bigl(x, y, \frac{dy}{dx}, \ldots, \frac{d^n y}{dx^n}\Bigr) = 0,$$

where G is a polynomial. Using Jacobi's techniques for bringing the equation into canonical form, she reduced this equation to a first-order system

$$\begin{array}{rcl} D_{n+2}G(x, y_0, y_1, \ldots, y_n)\frac{dy_1}{dx} & - & G_1(x, y_0, y_1, \ldots, y_n) = 0 \\ \ldots\ldots\ldots\ldots\ldots & & \ldots\ldots\ldots\ldots\ldots \\ D_{n+2}G_n(x, y_0, y_1, \ldots, y_n)\frac{dy_n}{dx} & - & G_n(x, y_0, y_1, \ldots, y-n) = 0, \end{array}$$

where $G_1, \ldots, G_n$ are polynomials. The method of majorants can then be applied to show that the power series generated assuming any initial values for $y_0(0), \ldots, y_n(0)$ actually has a positive radius of convergence.

Kovalevskaya's most general theorem concerned a partial differential equation

$$G\Bigl(x, x_1, \ldots, x_r, \varphi, \ldots, \frac{\partial^{\alpha+\alpha_1+\cdots+\alpha_r}\varphi}{\partial x^\alpha \partial x_1^{\alpha_1}\cdots\partial x_r^{\alpha_r}}, \ldots\Bigr) = 0,$$

where G is an irreducible polynomial in its variables and the maximum order of the derivatives is n, that is, $\alpha + \alpha_1 + \cdots + \alpha_r \le n$. Kovalevskaya described normal form as follows:

> I shall now consider the case, which I shall call *normal*, in which at least one of the derivatives $\frac{\partial^n\varphi}{\partial x^n}, \frac{\partial^n\varphi}{\partial x_1^n}, \ldots, \frac{\partial^n\varphi}{\partial x_r^n}$, say $\frac{\partial^n\varphi}{\partial x^n}$, actually occurs in G.

* In this same paper in which he introduced normal form Jacobi developed his "last multiplier method," which turns out to have an important connection with Kovalevskaya's other major papers.

The vague phrase "actually occurs" seems to mean that the equation can be *solved* for that derivative.

Under this assumption she was able to show that it was possible to find a function $\varphi(x, x_1, \ldots, x_n)$ that is analytic in a neighborhood of the point $(a, a_1, \ldots, a_n)$ and satisfies the equation and the initial conditions $\varphi(a, x_1, \ldots, x_r) = f_0(x_1, \ldots, x_r)$, $\frac{\partial\varphi}{\partial x}(a, x_1, \ldots, x_r) = f_1(x_1, \ldots, x_r), \ldots, \frac{\partial^{n-1}\varphi}{\partial x}(a, x_1, \ldots, x_r) = f_{n-1}(x_1, \ldots, x_r)$, provided the functions $f_0, \ldots, f_{n-1}$ are analytic on a neighborhood of $(a_1, \ldots, a_{n-1})$.

This result is now known as the *Cauchy–Kovalevskaya* theorem, and it remains to this day one of the fundamental results in this area. As with any mathematical result, it depends on clever and efficient use of results and techniques derived by earlier mathematicians. Kovalevskaya's main contribution, as I stated above, was to recognize the importance of normal form as a sufficient condition for the functioning of the method of majorants and hence for the validity of the method of undetermined coefficients.

In this paper Kovalevskaya unified a number of ideas of earlier mathematicians and brought clarity and systematization to an area that had previously lacked coherence. She provided a counterexample, showing that a problem might fail to have an analytic solution in the absence of normal form. She considered the one-dimensional heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},$$

with the *spatial* initial condition $u(x, 0) = \frac{1}{1-x}$. For this problem the method of undetermined coefficients generates the power-series

$$u(x, t) = \sum_{\nu=0}^{\infty} \frac{(2\nu)!}{\nu!} \frac{t^\nu}{(1-x)^{2\nu+1}},$$

which diverges for all $t \neq 0$.

A physicist might object that Kovalevskaya's initial temperature distribution becomes infinite at the point $x = 1$. However, the same result would be achieved with the initial condition $u(x, 0) = \frac{1}{1+x^2}$, which is physically realizable. The series generated by the method would diverge for all values of t except $t = 0$.

By contrast, since the equation is of order 2, Kovalevskaya's theorem guarantees that any prescribed temperature and temperature gradients at the point $x = 0$ that are analytic functions of time will yield an analytic temperature $u(x, t)$. The crucial property needed to get analytic solutions is being able to solve the differential equation for a pure derivative *of the same order as the equation*.

Weierstrass was astounded by this counterexample. He stated in his letter recommending the degree for Kovalevskaya that he had believed that the equations of mathematical physics would necessarily have analytic solutions. Apparently he was guided by physical intuition to a greater degree than the prevailing view of him would suggest. Analytic functions have a deterministic property that harmonizes very well with the possibility of physical prediction, as Lagrange had pointed out. (Knowing the values of a physical system over any finite interval of time makes it possible to predict the future state of the system for all time, provided the functions describing it are analytic.) Kovalevskaya had shown the presence of a previously unsuspcted subtlety in this determinism.

The rotation of a rigid body. Kovalevskaya's papers on the rotation of a rigid body about a fixed point represent an attempt to penetrate an unexplored region of mathematical physics using the resources of analytic function theory. The problem she was attempting to solve is a system of six ordinary differential equations. Her approach was a combination of the two approaches—power-series and closed-form—mentioned above. The way in which the two approaches were combined is interesting in itself. She first asked whether the equations could have meromorphic solutions, that is, solutions expressible as power series in time (including possibly a finite number of negative powers of time). That portion of the problem is the power-series approach. Two such cases were already known and had been studied in detail by Euler and Lagrange. Having determined the third and only new case in which such solutions exist, she set out to find the solution in closed form, thereby making use of the very latest results of mathematical research in the form of theta functions of two variables.

The differential equations of motion. Physicists know very well that the particular differential equations that describe a physical system are not obvious consequences of the great laws of physics such as conservation of energy or Newton's second law of motion. Every mathematical model results from a conscious decision to include certain properties and neglect others. For example, in the famous description of a vibrating string held fast at two points, one can represent the displacement of the string above the point with coordinate x at time t as a function $u(x, t)$ and derive the equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}.$$

This model assumes that the tension in the string is the same at all times and all points, that an individual point on the string moves only vertically, not horizontally, that all powers of the slope $\frac{\partial u}{\partial x}$ higher than the first can be ignored, and so forth. It requires considerable insight and a bit of luck to come up with a judicious selection of properties yielding a model that provides accurate predictions. With the passage of time, good models get reformulated and refined, as Newton's basic outline of the laws of motion was refined to include the concept of energy and given a more elegant mathematical formulation in Lagrange's exposition of mechanics, which was then further refined in the presentation of Hamilton.

So it was with the equations of motion of a rotating rigid body. It was Euler who first discovered that every solid body has a set of three mutually perpendicular axes such that the products of inertia of the body become zero in these axes. That is, if the axes are labeled x, y, and z, the body occupies a region $\mathcal{B}$, and the mass distribution of the body is dM, then

$$\int_{\mathcal{B}} yz\,dM = \int_{\mathcal{B}} zx\,dM = \int_{\mathcal{B}} xy\,dM = 0.$$

The motion of the body is most easily described by giving the nine direction cosines of these canonical body axes at any time t in terms of a set of mutually perpendicular axes fixed in space. (In modern terms, this amounts to giving the rotation matrix $R(t)$ that changes spatial coordinates into body coordinates.) To solve this problem for the case when no external torque acts on the body, Euler derived a set of nine differential equations, only the first three of which became a permanent addition to the literature on this problem. These three equations were

$$\begin{aligned} dx + \frac{c^2 - b^2}{a^2} yz\,dt &= 0, \\ dy + \frac{a^2 - c^2}{b^2} zx\,dt &= 0, \\ dz + \frac{b^2 - a^2}{c^2} xy\,dt &= 0. \end{aligned}$$

Here the quantities a^2, b^2 and c^2 are the moments of inertia of the body about its principal axes, that is,

$$a^2 = \int_{\mathcal{B}} (y^2 + z^2)\,dM, \quad b^2 = \int_{\mathcal{B}} (z^2 + x^2)\,dM, \quad c^2 = \int_{\mathcal{B}} (x^2 + y^2)\,dM.$$

The variables x, y, and z are the coordinates of a point on the body. Euler discussed in detail the motion of a body upon which no external torque is acting and showed that the direction cosines needed to describe the motion in this case can be expressed as elliptic integrals, reducing to elementary functions if two of the three principal moments of inertia are equal.

The fact that elliptic integrals arise even in the simplest case of the motion of a rigid body showed that the mathematics involved in describing such motion in general was likely to be very complicated. As Euler wrote,

> . . . when difficulties are. . . encountered because of the large number of variables, it is no longer in Mechanics that one should seek the means to overcome them, since it seems that the nature of such a motion is not susceptible of any simpler calculation. Everything then devolves onto the calculator, who must render the necessary support in analysis so as to solve the equations that determine the motion; but there is no doubt that an infinite number of cases exist that are absolutely unsolvable due to the limitations of analysis. . .

The Lagrange case. The next advance in the subject came with Lagrange's reformulation of Newtonian mechanics in terms of variational principles. Lagrange was able to study the general case of motion of a body having equal moments of inertia about two of its principal axes and its center of gravity on the third principal axis provided the external torque acting on the body results purely from gravitational attraction. To this day Lagrange's case is the one most frequently seen in practice. It was referred to in the nineteenth century as the case of a "heavy" body. In this case also, the solutions turned out to be expressible as elliptic functions.

The work of Jacobi. Once again the work of Jacobi appears as one of the key elements in Kovalevskaya's work. Although his influence does not show directly in the papers she wrote, it was crucial in setting the problem she solved in two different ways. First, his work on methods of solving equations in closed form had led him to a generalization of the technique that applies to exact differential equations. In general solving a system of $n-1$ ordinary differential equations of the form

$$\frac{dx_1}{X_1} = \frac{dx_2}{X_2} = \cdots = \frac{dx_n}{X_n},$$

requires finding $n-1$ independent "integrals," that is, functions $\varphi(x_1, \ldots, x_n)$ that are constant when the variables $x_1, \ldots, x_n$ satisfy the differential equation. For a system in which the divergence of the vector $(X_1, \ldots, X_n)$ vanishes, that is,

$$\frac{\partial X_1}{\partial x_1} + \cdots + \frac{\partial X_n}{\partial x_n} = 0,$$

and $n-2$ independent integrals have been found, Jacobi showed how to form the last independent integral by integrating an exact differential equation in two variables. This method became known as his "last-multiplier" method. The significance of this result for the problem of rotation of a rigid body will appear below.

Second, since Jacobi had shown that elliptic functions can be most elegantly expressed as quotients of theta functions of one variable, it followed implicitly that the solution of the problem of a rotating rigid body in the cases of Euler and Lagrange must be expressible in terms of these functions. Jacobi made this implicit conclusion explicit in an 1849 paper in which he wrote out the direction cosines of the body axes in terms of space axes using the two theta functions θ and H exhibited above. For example, the entry in the first row and first column of the rotation matrix $R(t)$ mentioned above is

$$\frac{\theta(K)\big[H(u+ia) + H(u-ia)\big]}{2H(K-ia)\theta(u)},$$

where u is a linear function of time ($u = nt + \tau$).

The importance of Jacobi's work was shown by the problem posed as the topic for the Leibniz Prize at the Prussian Academy of Sciences in 1852:

> It is known that the number of cases in which the differential equations of analytic dynamics can be integrated in closed form, or even reduced to quadratures, is quite limited; and in view of the repeated efforts that the greatest mathematicians have applied to this subject, it is very probable that most of the mechanical problems whose solution has not yet been achieved in this form are by their nature not susceptible of integration by quadratures and can be effectively handled only through the introduction of other analytic forms. Since Jacobi has recently given a beautiful representation in series form of the rotation of a rigid body on which no

accelerative force acts, it seems worthwhile to attempt to give a wider expression to the application of series and use them to handle cases of rotational motion that have not yet been reduced to quadratures. One such case is offered by the rotation of a heavy body, for which reduction to quadratures has been attained only in one special case due to Lagrange. The Academy therefore makes the complete solution of this problem the subject of a competition and poses the problem:

> "Integrate the differential equations for the motion of a body rotating about a fixed point on which the only accelerative force is that of gravity, by means of regular series representing as explicit functions of time all quantities required to determine the motion."

Despite the prize of 100 ducats, however, no entries were received in this competition by the deadline of 1855. The competition was extended for another three years, again without any entries.

Weierstrass and C. Neumann. Since Weierstrass was in Berlin at this time, he cannot have been unaware of this competition. From a mathematical point of view the problem posed for the competition would have been particularly attractive to him at that time, since he had been one of the pioneers in using the newly-discovered theta functions of two variables to solve the general Jacobi inversion problem. It undoubtedly seemed to him that the time had come to make new advances in the problem of rigid body motion. There were other advances along these lines as well that would have convinced him that this problem could now be solved, the most important of which was the dissertation of Carl Neumann (1832–1925). The problem Neumann posed was *to describe the motion of a point constrained to move on the unit sphere* $x^2 + y^2 + z^2 = 1$ *and subject to a force whose potential is* $u(x, y, z) = ax^2 + by^2 + cz^2$, where a, b, and c are real numbers (not necessarily positive). Neumann found four possible cases of motion, depending on the values of the initial conditions imposed on the motion. Most importantly, he found that by using the Hamilton–Jacobi formulation of the equations of motion, parameterizing the sphere by a pair of coordinates λ_1, λ_2, he obtained the following equations for the trajectory of a solution

$$\varepsilon_1 \int_{l_1}^{\lambda_1} \frac{d\lambda}{L} + \varepsilon_2 \int_{l_2}^{\lambda_2} \frac{d\lambda}{L} = 0, \qquad \varepsilon_1 \int_{l_1}^{\lambda_1} \frac{\lambda\, d\lambda}{L} + \varepsilon_2 \int_{l_2}^{\lambda_2} \frac{\lambda\, d\lambda}{L} = t,$$

where $L = \sqrt{8(\lambda - a)(\lambda - b)(\lambda - c)(\lambda - A)(\lambda - B)}$, A and B are arbitrary constants, $\varepsilon_1 = \pm 1$ and $\varepsilon_2 = \pm 1$. (The four possible values of the pair $(\varepsilon_1, \varepsilon_2)$ give the four cases of motion distinguished by Neumann.) These are, of course, *exactly* the equations of the Jacobi inversion problem! Naturally Neumann was able to express the solutions in terms of theta functions of two variables.

Weierstrass was aware of Neumann's work and no doubt intrigued by the application of theta functions of two variables. He must have seen this area as a fruitful one for further investigation. As we shall see below, he suggested this line of work to Kovalevskaya.

The mathematical mermaid. It is time to return to Jacobi's other contribution to this problem, namely his development of the last-multiplier method of solving a system of ordinary differential equations with zero divergence. The final form of the Euler equations was given by the British mathematician R. B. Hayward in 1858. For a body subject only to gravitational torque having one point fixed at (0, 0, 0) with center of gravity at the point (x_0, y_0, z_0) in a set of mutually perpendicular axes fixed in the body, the system consists of the following six equations for six unknown functions $p(t)$, $q(t)$, $r(t)$, $\gamma_1(t)$, $\gamma_2(t)$, $\gamma_3(t)$ which are respectively the three coordinates of the angular velocity vector of the body and the three coordinates (in body coordinates) of a unit vector pointing downward in space.

$$\begin{aligned} A\frac{dp}{dt} &= (B - C)qr + Mg(y_0\gamma_3 - z_0\gamma_2), & \frac{d\gamma_1}{dt} &= r\gamma_2 - q\gamma_3, \\ B\frac{dq}{dt} &= (C - A)rp + Mg(z_0\gamma_1 - x_0\gamma_3), & \frac{d\gamma_2}{dt} &= p\gamma_3 - r\gamma_1, \\ C\frac{dr}{dt} &= (A - B)pq + Mg(x_0\gamma_2 - y_0\gamma_1), & \frac{d\gamma_3}{dt} &= q\gamma_1 - p\gamma_2. \end{aligned}$$

Here A, B, and C are the three principal moments of inertia (which Euler had denoted a^2, b^2, c^2), M is the mass of the body, and g is the acceleration of gravity. For purposes of analysis, it is important to observe that they really give six different expressions for dt, none of which involves time. That is, they can be written as the set of five equations

$$\frac{dp}{P} = \frac{dq}{Q} = \frac{dr}{R} = \frac{d\gamma_1}{\Gamma_1} = \frac{d\gamma_2}{\Gamma_2} = \frac{d\gamma_3}{\Gamma_3},$$

where none of the equations contains t, and each of the denominators is a (quadratic) polynomial in the six variables. Even more convenient is the fact that the polynomial P is independent of p, Q is independent of q, and so forth, that is, each denominator is independent of the variable that occurs in the differential of its numerator. That means certainly that the divergence of the system is zero. Hence the Jacobi last-multiplier method will yield a complete set of five independent integrals provided we can find four independent integrals.

Now three integrals are immediate: (1) Conservation of energy shows that the function $\frac{1}{2}(Ap^2 + Bq^2 + Cr^2) + Mg(x_0\gamma_1 + y_0\gamma_2 + z_0\gamma_3)$ must be constant. (2) Since the force is vertical, the torque is horizontal, and hence the vertical component of the angular momentum must be constant, that is, $Ap\gamma_1 + Bq\gamma_2 + Cr\gamma_3$ must be constant. (3) Since $(\gamma_1, \gamma_2, \gamma_3)$ is a unit vector, $\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1$. Thus the general problem would be solved if only one more independent integral could be found. Where should one look for it? No one knew at the time. That is why the problem was so tempting, and yet so elusive, that the Germans referred to as *die mathematische Nixe* (the mathematical mermaid).

Kovalevskaya's work. Kovalevskaya worked on this problem while she was studying with Weierstrass, but she was not able to make any progress on it. It was while she was trying to re-establish her mathematical credentials, in the early 1880s that she found herself returning to this problem with renewed hope. In November 1881 she wrote to Mittag-Leffler,

> This past autumn I began to work on the integration of the partial differential equations that arise in optics in the problem of refraction of light in a crystalline medium. This work was quite well along when I had the weakness to allow myself to be distracted by another question, which has never ceased to rattle around in my head since almost the beginning of my mathematical studies, and in which, for a time, I feared I would find myself surpassed by others. The problem involves solving the general case of rotation of a heavy body about a fixed point by means of abelian functions. M. Weierstrass had once previously suggested that I work on this problem, but all my attempts at the time were fruitless, and M. Weierstrass' own research showed that the differential equations of this problem cannot be satisfied by single-valued functions of time. This result compelled me to abandon this problem for a while. But since then the beautiful, still-unpublished, research of our master on the stability of the solar system and the analogy with other problems of dynamics have renewed my zeal and given me the hope of satisfying the conditions of this problem by abelian functions whose arguments are nonlinear functions of time. This research seemed so interesting and beautiful to me that I forgot everything else for a while and abandoned myself to it with all the impetuosity of which I am capable. The route I followed consisted in expressing the variables of the problem by theta functions of two variables that for certain values of the constants reduce to the elliptic theta functions that arise in the special case of Lagrange, then trying to choose them so as to be able to integrate the differential equations between the theta functions and time. the calculations which I got into in this way were so difficult and complicated that I cannot yet say if I will reach the desired end by this route. in any case I hope to know in two or three weeks at the most what I should do about it, and M. Weierstrass is consoling me that even in the worst case I could always invert the question and try to find out which forces lead to a rotation whose variables can be expressed by abelian functions—a poor problem, to be sure, and one far from having the same interest as the one I have set myself, but one which I shall have to settle for if I have bad luck, relying on the example of M. Neumann, who chose an analogous problem for his doctoral dissertation.

Working her ideas into a substantial result took Kovalevskaya another four years. Only in June 1886 was the work sufficiently advanced to present to Hermite and his colleagues in Paris. As is well-known,

A letter from Kovalevskaya, probably to Hermite, communicating her discovery of a new case in which the equations of motion of a rigid body can be integrated in finite form. Courtesy Institut Mittag-Leffler.

the Paris Academy, at the instigation of Hermite, arranged a prize competition suitable for this work, which Kovalevskaya entered and won in 1888.

The result: Part I. The general equations of motion of a heavy body about a fixed point turned out to be too complicated to have meromorphic solutions (as Kovalevskaya indicates Weierstrass had already shown, though he never published this work). Taking as the basic problem the system of five equations, Kovalevskaya posed the problem of the circumstances under which such a system could by satisfied by a system of six meromorphic functions of time $p(t)$, $q(t)$, $r(t)$, $\gamma_1(t)$, $\gamma_2(t)$, $\gamma_3(t)$. Thus she assumed the system of equations

$$\begin{aligned}
p(t) &= t^{-n_1}(p_0 + p_1 t + p_2 t^2 + \cdots),\\
q(t) &= t^{-n_2}(q_0 + q_1 t + q_2 t^2 + \cdots),\\
r(t) &= t^{-n_3}(r_0 + r_1 t + r_2 t^2 + \cdots),\\
\gamma_1(t) &= t^{-m_1}(f_0 + f_1 t + f_2 t^2 + \cdots),\\
\gamma_2(t) &= t^{-m_2}(g_0 + g_1 t + g_2 t^2 + \cdots),\\
\gamma_3(t) &= g^{-m_3}(h_0 + h_1 t + h_2 t^2 + \cdots).
\end{aligned}$$

The equations themselves immediately yield the fact that $n_1 = n_2 = n_3 = 1$ and $m_1 = m_2 = m_3 = 2$, that is, the components of the angular velocity have at worst a simple pole, and those of the unit vertical vector at worst a pole of order two.

If the general solution of the system is to be expressed in this way, five of the coefficients of the power series must be arbitrary. The equations provide a recursive system of linear equations for each set of coefficients. In order to get the required five degrees of freedom, the complete set of these systems must have a rank deficiency of five. Kovalevskaya, overlooking the case in which one of the systems might have a rank deficiency greater than one, examined the cases in which five different systems each have rank deficiency one, and found that this is possible only if $A = B = 2C$ and $z_0 = 0$. That is the *Kovalevskaya case* of motion. It means that the body has partial kinetic symmetry and that its center of gravity lies in the plane through the fixed point spanned by the two principal axes about which the moments of inertia are equal. (To exhibit an example of a body having these properties Kovalevskaya appealed to Hermann Amandus Schwarz, another student of Weierstrass, who produced a model consisting of a pair of identical cylinders with parallel axes.) Thus, Kovalevskaya had carried out the modest goal Weierstrass had suggested, having discovered the last case in which the equations of the problem can be expressed as meromorphic functions of time.

Bypassing the last-multiplier method. After discovering this case, Kovalevskaya also discovered along the way a fourth algebraic integral. Normalizing the units so that $A = B = 2$, $C = 1$, and choosing axes in the plane where all moments of inertia are equal in such a way that the center of gravity lies on one of the axes, one gets $y_0 = 0$ also. For this case the fourth algebraic integral can be written in complex form as

$$\big[(p+qi)^2 + Mgx_0(\gamma_1 + i\gamma_2)\big]\big[(p-qi)^2 + Mgx_0(\gamma_1 - i\gamma_2)\big].$$

At this point, *in principle*, Jacobi's last-multiplier method provides a recipe to follow to construct the fifth independent integral, after which the trajectories could be made to appear as (in modern terms) the intersections of the level surfaces of these five functions of the six variables. One can well imagine, however, that the computations involved are going to be very complicated. In fact, Kovalevskaya does not follow this route, nor does she even mention the last-multiplier method. Instead she resorted to an intricate series of changes of variable in order to get back, in the end, to the same Jacobi inversion problem that Neumann had derived. Her system was in fact simply the differentiated form of Neumann's system:

$$0 = \frac{ds_1}{\sqrt{\mathcal{R}(s_1)}} + \frac{ds_2}{\sqrt{\mathcal{R}(s_2)}}, \quad dt = \frac{s_1\,ds_1}{\sqrt{\mathcal{R}(s_1)}} + \frac{s_2\,ds_2}{\sqrt{\mathcal{R}(s_2)}},$$

in which $\mathcal{R}(s)$ is a polynomial of degree 5.

The result: Part 2. Complicated as this work was, it was only the beginning of Kovalevskaya's work. The expression of the variables s_1 and s_2 as quotients of theta functions of two variables whose arguments are linear functions of time was by 1886 a standard procedure. It took Kovalevskaya 16 pages to get this far on the problem. The true depth of the problem lies in the recovery of the original variables from the dazzling series of changes of variables mentioned above. That part of the work required 50 more pages of very discouraging computation. In the end she was able to express the components of the angular velocity vector as quotients of the following form:

$$p = -i\,\frac{L_1P_1 + M_1P_2 + N_1P_3}{LP_1 + MP_2 + NP_3},$$

where L, M, N, L_1, M_1, N_1 are constants determined algebraically from the data of the problem, and P_1, P_2, and P_3 are quotients of theta functions of two variables, each of which is a linear function of time.

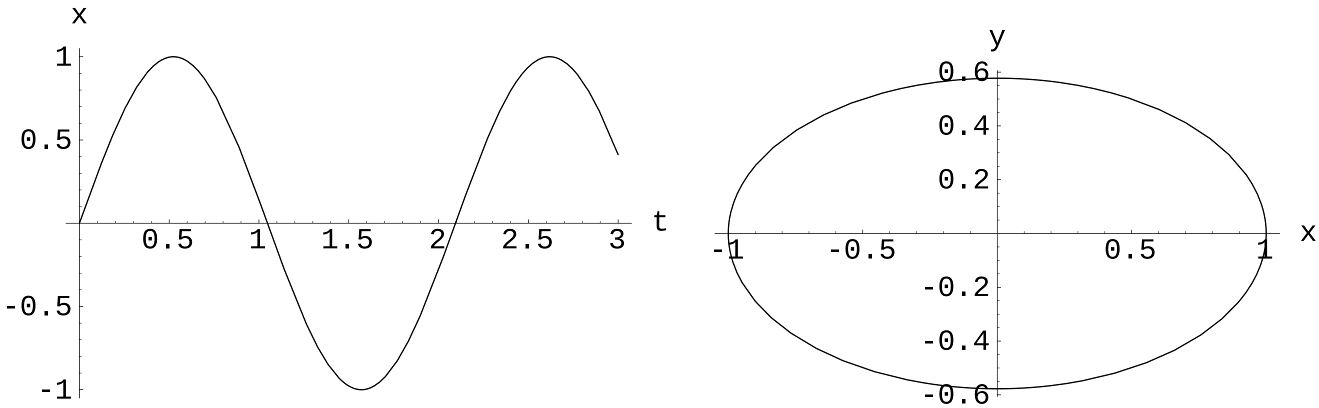
Even our abbreviated sketch of the paper shows how very complicated the motion of a rigid body is, even for a body possessing the considerable amount of symmetry of the Kovalevskaya case. The analytic description of the general case must necessarily be at least this complicated, and probably far worse. Kovalevskaya completed her work by showing that the components of the rotation matrix that describes the position of the body axes in terms of spatially fixed axes can all be expressed using quotients of theta functions of two variables, each of which is a linear function of time.

Impact of this work in its own time. The report of the jury for the 1888 Bordin Prize showed the aspect of the work that contemporaries considered to be of most importance:

> The author has done more than merely adding a result of very high interest to those bequeathed to us by Euler and Lagrange; he has made a profound study of his result in which the resources of the modern theory of theta functions of two independent variables make it possible to give the solution in the most precise and elegant form. The result is a new and memorable example of a mechanical problem involving these transcendental functions, whose applications had previously been confined to pure analysis and geometry. . .

This point of view was shared by Weierstrass and Hermite. There seemed to be a hope that new advances could be made in mathematical physics by solving more complicated differential equations and that the theory of abelian functions would play an important role in this advance.

Lasting significance of the work. As matters turned out, this hope was postponed for a very long time. Except for Neumann's dissertation, which Weierstrass called a "cleverly contrived problem," and Kovalevskaya's result, there were no uses of abelian functions in general or theta functions in particular until the early 1970s.



The displacement of the end of a spring as a function of time, and the phase portrait of the motion.

Then B. A. Dubrovin, V. B. Matveev, S. P. Novikov, H. McKean, and P. Moerbeke, proved some results on solutions of the Korteweg–de Vries equation and Hill's equation, which are used to describe wave motion in a channel and lunar motion respectively. Dubrovin, Matveev, and Novikov showed that the potential for a stationary higher-order Korteweg–de Vries equation can be expressed using theta functions of several variables. McKean and Moerbeke showed that Hill's equation is contained in the Neumann equations (so that this problem is not so "contrived" as Weierstrass believed).

Even so, the applications of abelian functions in mathematical physics remain very limited. Interest in the Kovalevskaya gyroscope, however, had a recent revival, and the inspiration for the renewed interest comes from an approach to differential equations that Kovalevskaya could not have known about in 1888, as its foundations were only then being laid by her friend Henri Poincaré. (At the time Poincaré was using his new approach to differential equations to study another important physical problem, the three-body problem. By coincidence he received a prize from the Swedish King Oscar II for that work in a competition arranged by Mittag-Leffler, with Kovalevskaya's collaboration.)

Modern mathematicians tend to study systems of differential equations either geometrically, in terms of the phase portrait of the system, or asymptotically, by constructing a divergent series whose pth partial sum approximates the solution to order $p + 1$ for large values of the independent variable. The geometric approach has proved well-suited to the Kovalevskaya case. To explain it, we look at the simple equation of an undamped vibrating spring, where $x(t)$ represents the displacement of the end of the spring from its equilibrium position at time t:

$$x''(t) + \omega^2 x(t) = 0.$$

Although the complete solution of this equation is known to every undergraduate mathematics major ($x(t) = A\cos(\omega t) + B\sin(\omega t)$), an indirect way of studying it is to introduce a pair of variables $\bigl(x(t),\, y(t)\bigr)$ depending on t, where $y(t) = x'(t)$. Because of the differential equation and the fact that $y(t) = x'(t)$, this pair satisfies the system of equations

$$x'(t) = y(t), \quad y'(t) = -\omega^2 x(t).$$

This being a first-order equation, it is easy to draw the field of vectors tangent to its solutions, from which the trajectories traced out by the point $\bigl(x(t),\, y(t)\bigr)$ can be drawn. The (x, y)-plane is partitioned into disjoint trajectories of this system, which constitute what is called the *phase portrait* of the system. In fact it is easy to see that $\omega^2 x(t)x'(t) + y(t)y'(t) = 0$, so that $\omega^2\bigl(x(t)\bigr)^2 + \bigl(y(t)\bigr)^2 = C$. Hence the phase trajectories form a family of ellipses with center at the origin whose axes are in the ratio $\omega : 1$. In interpreting the phase portrait it is important to remember that its trajectories are *not* a picture of the motion of a particle belonging to the physical system. Points on a trajectory represent the *state* of the system at different instants of time, and moving around a trajectory corresponds to continuous changes of state. In the present case, the state of the system is described by its displacement and velocity. The fact that a trajectory is an ellipse means that the displacement and velocity are oscillating at the same frequency, resulting in a periodic motion.

The phase portrait for the Kovalevskaya gyroscope is exceedingly complicated. (It lies in six-dimensional real space.) The significance of the result lies precisely in the fact that this phase space can be described completely, despite its complexity. A thorough analysis of the space has been carried out by H. R. Dullin, P. H. Richter, and A. P. Veselov. This analysis led to the production of an extremely elegant computer-graphic description of the phase portrait for the Kovalevskaya case, accompanied by a film of a physical model. This film is available from the Institut für den Wissenschaftlichen Film. (Postfach 2351, 37013 Göttingen. Tel. 49/551/15024-0. e-mail iwf-goe@iwf.de, http://www.iwf.gwdg.de. "Kovalevskaya Top," Film No. C1961.). An earlier analysis of this case was carried out by Jean-Pierre Françoise, as an example of a completely integrable Hamiltonian system.

Minor works. In addition to her work on the two topics discussed in detail above, Kovalevskaya published four other mathematical papers, three of which date to her student years in Berlin and one of which was from the early 1880s. Although one of these, on hyperelliptic integrals that degenerate to elliptic integrals, was an elegant piece of work that impressed Poincaré, none of them would be remembered today as important contributions to nineteenth-century analysis. We shall therefore describe each of them in a few sentences.

Degenerate abelian integrals. There are certain values of the parameter in a hyperelliptic integral that cause it to become an elliptic integral. Weierstrass had discovered that such would be the case if the theta functions associated with the integral had a certain form. This criterion was a "transcendental" condition, since theta functions are not themselves algebraic functions. Kovalevskaya found an algebraic criterion based on the location of the points where the curve associated with the integral has double tangents. This paper lay unpublished for ten years until Kovalevskaya succumbed to the "publish or perish" ambience of the day and published it in the *Acta Mathematica* in 1884. There it attracted the attention of Poincaré, who found it sufficiently impressive to publish his own extensions of it.

The shape of Saturn's ring. This paper occupies an anomalous position among Kovalevskaya's works. It does not seem to be mentioned in any of the letters she received from Weierstrass as a student. The paper was based on Laplace's study of a ring orbiting a planet for which the shape of a cross-section was determined by assuming that a thin layer of liquid on the surface would be in equilibrium. (Not that the ring was actually believed to be liquid. Rather this assumption was a way of saying that no shear stresses were acting on the ring as a result of the gravitational attraction of the planet.) Laplace had assumed an elliptic cross-section of the ring. Kovalevskaya attempted to generalize this assumption by taking the cross section in polar coordinates as the solution of the equation

$$r(\theta) = m_0 + m_1 \cos\theta + m_2 \cos 2\theta + \cdots,$$

Truncating the series by setting $m_2 = m_3 = \cdots = 0$ yields the Laplace case. Kovalevskaya carried the series out to one more term, obtaining an oval shape. This is the only paper in which she made serious efforts at numerical approximation. Like the paper on abelian integrals, this one formed part of her dissertation and lay unpublished for a decade until her friend the astronomer Hugo Gyldén published it for her in the *Astronomische Nachrichten*.

The Lamé equations. When Kovalevskaya was trying to re-establish her credentials as a mathematician in 1881, Weierstrass gave her a paper he had written many years earlier containing a surface-integral method of solving certain differential equations. He suggested that she apply this technique to solve the differential equations that Gabriel Lamé (1795–1870) had derived for propagation of a wave in a solid medium. Lamé had hoped to explain light as an elastic disturbance in a medium, but the solutions of his equations could not explain propagation of light from a point source, since the solutions became infinite at the source. Weierstrass hoped that his technique would yield new solutions that did not become infinite at that point. (A philosophical issue hung on the outcome, as Lamé had been driven to the point of introducing a weightless ether in order to avoid rejecting the application of his beloved elasticity theory in the explanation of light.) Despite enormous personal hardship at the time, Kovalevskaya produced formulas for a solution. Weierstrass, however, wasn't feeling well at the time and delayed his consent to publication of his portion of the paper, which was of extreme importance to Kovalevskaya. In the end, he allowed Kovalevskaya to publish the paper in the *Acta Mathematica* with quotation marks around the portion of it due to him. Although the paper was highly regarded and was proofread by no less a mathematician than Carl Runge (1856–1927), apparently no one attempted to apply it to any particular problem until Vito Volterra (1860–1940) did so in early 1891, just at the time of Kovalevskaya's death. Reading this paper in preparation for giving a course on light propagation, Volterra discovered that in fact Kovalevskaya's formulas do not satisfy the Lamé equations. Thus it was necessary for Mittag-Leffler to print a retraction of this paper simultaneously with his obituary of Kovalevskaya.

Bruns' theorem. E. H. Bruns (1848–1919) is best known for having proved that the differential equations of the three-body problem have only a ten-dimensional space of algebraic integrals. He also proved that the potential of a body bounded by a surface $W(x, y, z) = 0$, where W is an analytic function, is analytic at each regular point of the surface (each point where the gradient of W does not vanish). The proof of this result involves constructing an analytic solution of a boundary-value problem for Poisson's equation in the region. Kovalevskaya derived this result as a corollary of the Cauchy–Kovalevskaya theorem. There seems to be little doubt that this paper was written while she was working on her dissertation; Weierstrass alludes to such a paper in his letter requesting her degree. However, she did not decide to publish it until 1889, when, no doubt exhausted from the work on the rotation of a heavy body, she needed a paper to give at a conference. Mittag-Leffler published it after her death in the *Acta Mathematica*.

Kovalevskaya as a mathematician. How significant was Kovalevskaya as a nineteenth-century mathematician? Would she be remembered today if she had not been the first woman professor of mathematics (or any other subject) since the Renaissance? The answer to the second question is a definite yes: The Cauchy–Kovalevskaya Theorem remains to this day a result of permanent value, and she advanced that topic further than anyone before her. The rotation of a heavy body, while an excellent piece of work, did not have a comparable impact. Based purely on her mathematical work, she would occupy a respectable place in the elite club of nineteenth-century European mathematicians. (This was, of course, a large club, with several hundred members.) One cannot, of course, forget the obstacles that she had to overcome, nor should one overlook her conscious efforts to provide to other women the same opportunities for which she struggled. In the present essay I have not mentioned these efforts, but even when that important facet of her life is ignored, she remains an impressive figure. Mittag-Leffler, who did so much to assist her, may be the best person to bring the present essay to a close and render a judgment on Kovalevskaya's life as a whole:

> . . . it is perhaps neither as a mathematician nor writer that one should properly appreciate or judge this woman of so much spirit and originality. As a person she was even more remarkable than one would judge from her works. All those who knew her and were near to her, to whatever circle or part of the world they belonged, will remain forever under the lively and powerful impression which her personality produced.

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