NAME:

Math 22 Spring 2009—QUIZ 3

1. A force of 30 N is required to maintain a spring stretched from its natural length of 12 cm to a length of 15 cm. How much work (in Joules) is done in stretching the spring from 12 cm to 22 cm?

Solution: Let x denote the distance the spring is stretched from its natural length. By Hooke's Law, $F = kx$, where k is the spring constant of the spring. Since 30 N is required to stretch the spring 3 cm, we have $30N = k(3\text{cm})$, so $k = 10\text{N/cm}$. Then the work required to stretch the spring from 12 cm to 22 cm is given by

$$W = \int_0^{10} (10\text{N/cm})x\,dx = (10\text{N/cm})\frac{x^2}{2}\Big|_0^{10\text{cm}} = (10\text{N/cm})(50\text{cm}^2) = 500\text{N-cm} = 5\text{N-m} = 5\text{J}.$$

2. Evaluate the following integrals:

(a) $\int p^5 \ln p\, dp$

Solution: Let $u = \ln p$, so $du = (1/p)dp$ and let $dv = p^5 dp$, so $v = (1/6)p^6$. Then

$$\int p^5 \ln p\, dp = (1/6)p^6 \ln p - \int (1/6p^6)(1/p\, dp) = (1/6)p^6 \ln p - (1/6)\int p^5\, dp$$
$$= (1/6)p^6 \ln p - (1/36)p^6 + C.$$

(b) $\int xe^{-x}\, dx$

Solution: Let $u = x$, so $du = dx$ and let $dv = e^{-x}dx$, so $v = -e^{-x}$. Then

$$\int xe^{-x}\, dx = -xe^{-x} - \int (-e^{-x})\, dx = -xe^{-x} - e^{-x} + C.$$

(c) $\int \cos\sqrt{x}\, dx$

Solution: First make a substitution: $x = t^2$. Then $dx = 2t\,dt$ and the integral becomes

$$\int \cos\sqrt{x}\, dx = \int \cos t(2t\,dt) = 2\int t\cos t\, dt.$$

Now integrate by parts, with $u = t$, so $du = dt$, and $dv = \cos t\, dt$, so $v = \sin t$. Then

$$2\int t\cos t\, dt = 2\left[t\sin t - \int \sin t\, dt\right] = 2t\sin t + 2\cos t.$$

Finally, substituting $t = \sqrt{x}$ gives

$$\int \cos\sqrt{x}\, dx = 2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x} + C.$$

(d) $\int \arcsin x \, dx$

Solution: Let $u = \arcsin x$, so $du = \frac{1}{\sqrt{1-x^2}} dx$, and $dv = dx$, so $v = x$. Then

$$\begin{aligned}\int \arcsin x \, dx &= x \arcsin x - \int x \frac{1}{\sqrt{1-x^2}} dx \\ &= x \arcsin x + \frac{1}{2} \int \frac{-2x}{\sqrt{1-x^2}} dx \\ &= x \arcsin x + \frac{1}{2} \int (1-x^2)^{-1/2} (-2x) dx \\ &= x \arcsin x + \frac{1}{2} \frac{(1-x^2)^{1/2}}{1/2} \\ &= x \arcsin x + \sqrt{1-x^2} + C.\end{aligned}$$

3. Evaluate the following integrals:

(a) $\int \sin^3 x \cos^2 x \, dx$

Solution:

$$\begin{aligned}\int \sin^3 x \cos^2 x \, dx &= \int \sin x (1 - \cos^2 x) \cos^2 x \, dx \\ &= -\int \cos^2 x (-\sin x) \, dx + \int \cos^4 x (-\sin x) \, dx \\ &= -\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} + C.\end{aligned}$$

(b) $\int_0^{\pi/2} \sin^2 x \, dx$

Solution: The average value of $\sin^2 x$ (or $\cos^2 x$) over a full half-period is 1/2, so the integral is $(1/2)(\pi/2) = \pi/4$.

Alternatively, use the half-angle formula: $\sin^2 x = \frac{1 - \cos 2x}{2}$. Then

$$\int_0^{\pi/2} \sin^2 x \, dx = \int_0^{\pi/2} \frac{1 - \cos 2x}{2} \, dx = \frac{x}{2} - \frac{\sin 2x}{4} \bigg|_0^{\pi/2} = \pi/4.$$

(c) $\int \tan^2 x \, dx$

Solution:

$$\int \tan^2 x \, dx = \int (\sec^2 x - 1) \, dx = \tan x - x + C.$$

(d) $\int (\tan^2 x + \tan^4 x) \, dx$

Solution:

$$\int (\tan^2 x + \tan^4 x) \, dx = \int \tan^2 x (1 + \tan^2 x) \, dx = \int \tan^2 x \sec^2 \, dx = \frac{\tan^3 x}{3} + C.$$