

**A Note on Strongly Closed Subgroups:
Generalizations of Theorems of Goldschmidt and Burnside**

Richard Foote

Department of Mathematics and Statistics, University of Vermont

1. Introduction

This note derives a consequence of [FIFo]: a generalization of a result of David Goldschmidt, [Go, Theorem A], and an observation on the classical N/C -Theorem of Burnside (see [Gr, Theorem 7.4.4]). Let G be a finite group, let p be any prime and let S be a p -subgroup of G . If $S \leq P$ for some subgroup P of G , we say S is *strongly closed* in P with respect to G if for every $s \in S$, whenever $s^g \in P$ for some $g \in G$ then $s^g \in S$. It is easy to verify that S is strongly closed in a Sylow p -subgroup P if and only if it is strongly closed in $N_G(S)$, so this notion of strong closure of S does not depend on the Sylow subgroup containing it, and we simply say S is strongly closed. Recall that $O_{p'}(G)$ is the largest normal subgroup of G of order prime to p , and S^G is the set of G -conjugates of S . The main theorem gives a powerful (local) condition for a finite group to be (globally) directly decomposable.

Theorem. *Let G be a finite group, let p be any prime and assume G contains strongly closed p -subgroups S_1 and S_2 such that $S_1 S_2 = S_1 \times S_2$. Let $K_i = \langle S_i^G \rangle$ for $i = 1, 2$ and let overbars denote passage to $G/O_{p'}(G)$. Then $\overline{K_1 K_2} = \overline{K_1} \times \overline{K_2}$.*

As a consequence of this, suppose a group G has an abelian Sylow p -subgroup A and, by Schur's Theorem, let H be a p' -Hall complement to A in $N_G(A)$ (or just let $H = N_G(A)/A$). It follows from the Fitting Lemma and Maschke's Theorem, (see [Gr, Theorem 5.2.2] and induction) that A decomposes under the action of H as

$$A = A_1 \times A_2 \times \cdots \times A_r \tag{*}$$

where each A_i is H -invariant and homocyclic with H acting irreducibly (possibly trivially) on each Frattini quotient group $A_i/\phi(A_i)$ (or equivalently on $\Omega_1(A_i)$) for all i . The subgroups A_i in such a decomposition need not be unique, but the Krull-Schmidt Theorem guarantees that they are unique up to H -module isomorphism, counting multiplicities of each type. The proof of Burnside's N/C -Theorem shows that if A_1, \dots, A_s are all the factors on which H acts trivially, then the commutator subgroup G' has Sylow p -subgroup $A_{s+1} \cdots A_r$.

A well-known, elementary fusion result used in the proof of Burnside's Theorem is that elements of A are conjugate in G if and only if they are conjugate in $N_G(A)$. This implies each A_i is strongly closed in G . The Theorem and induction thus immediately provide a generalization of Burnside's Theorem:

Corollary. *Let G be any finite group with an abelian Sylow p -subgroup A . Assume G has no non-trivial normal subgroup of order prime to p , and G is generated by its p -elements (i.e., $O_{p'}(G) = 1$ and $G = O_{p'}(G)$). If (*) is a decomposition of A under the action of $N_G(A)$, then G correspondingly decomposes as*

$$G = G_1 \times G_2 \times \cdots \times G_r$$

where A_i is a Sylow p -subgroup of G_i for each i . In particular, G is indecomposable if and only if $N_G(A)$ acts irreducibly on $\Omega_1(A)$.

Remarks:

It is important to note that S_1S_2 need not be Sylow in G , nor is S_i necessarily a Sylow p -subgroup of K_i ; so the Theorem generalizes Goldschmidt's result even for $p = 2$.

The corollary applies to the section $O_{p'}(G/O_{p'}(G))$ of an arbitrary finite group G possessing an abelian Sylow p -subgroup. This generalization of Burnside's Theorem follows from the main theorem, which in turn depends on the Classification of Finite Simple Groups. However, one can invoke the Classification directly, rather than via the lengthy [FlFo], to characterize all groups satisfying those hypotheses. Following that tack one reduces easily to where each G_i above is either a cyclic p -group or a simple group (since $G = O_{p'}(G)$, by properties of the known simple groups there are no outer automorphisms in G acting on these). Also, by a result of Gorenstein–Lyons [FlFo, Proposition 2.5], $N_{G_i}(A_i)$ acts irreducibly on $\Omega_1(A_i)$ for all i (and trivially, when $G_i = A_i$). These reductions are inherent in the proof of the main results of [FlFo] where more general configurations are likewise analyzed, and the properties of simple groups just cited are crucial to the general proofs; so it is safer to assert that the Burnside generalization is a corollary to the methods in [FlFo] rather than a novel outcome of the theorem herein.

In light of the above observations, if G is a finite group with an abelian Sylow p -subgroup A and $G = O_{p'}(G)$, then we obtain a factorization of $\overline{G} = G/O_{p'}(G)$ into a direct product of an abelian p -group — which is $Z(\overline{G})$, and equal to $\overline{A}_1 \cdots \overline{A}_s$ in the notation of Burnside's Theorem — and factors that are non-abelian simple groups. The latter factors are unique — they are the components of \overline{G} — and so their Sylow p -subgroups determine *unique* strongly closed (i.e., $N_G(A)$ -invariant) direct factors A_i of A .

Notation from [FlFo]:

In general let R be any p -subgroup of G . If M_1 and M_2 are normal subgroups of G with $R \cap M_i \in \text{Syl}_p(M_i)$ for both $i = 1, 2$, then $R \cap M_1M_2$ is a Sylow p -subgroup of M_1M_2 . Thus there is a unique largest normal subgroup N of G for which $R \cap N \in \text{Syl}_p(N)$; denote this subgroup by $\mathcal{O}_R(G)$. Thus

$$R \text{ is a Sylow } p\text{-subgroup of } \langle R^G \rangle \text{ if and only if } R \leq \mathcal{O}_R(G).$$

Note that $O_{p'}(G/\mathcal{O}_R(G)) = 1$; in particular, if $R = 1$ is the identity subgroup then $\mathcal{O}_1(G) = O_{p'}(G)$. In general, $R\mathcal{O}_R(G)/\mathcal{O}_R(G)$ does not contain the Sylow p -subgroup of any nontrivial normal subgroup of $G/\mathcal{O}_R(G)$; in other words, $\mathcal{O}_{\overline{R}}(\overline{G}) = 1$, where overbars denote passage to $G/\mathcal{O}_R(G)$. Throughout this note we freely use the observation that strong closure passes to quotient groups (cf. [Go, Lemma 2.2]).

Proof of the Theorem:

We may clearly assume

$$O_{p'}(G) = 1 \quad \text{and so} \quad O_{p'}(K_i) = 1 \quad \text{for } i = 1, 2.$$

It follows easily from strong closure and Sylow's Theorem that

$$K_i = \langle S_i^{K_i} \rangle \quad \text{for } i = 1, 2. \tag{1}$$

Thus we may assume

$$G = K_1K_2. \tag{2}$$

It suffices to show $K_1 \cap K_2$ has order prime to p .

Let $N_i = \mathcal{O}_{S_i}(G)$ for $i = 1, 2$. If $S_i \leq N_i$ for both $i = 1, 2$, then as $S_i \in \text{Syl}_p(N_i)$ it follows that $N_1 \cap N_2 \leq \mathcal{O}_{p'}(G) = 1$. Furthermore, $K_i \leq N_i$ and so the theorem is true in this case. We may therefore choose notation to assume that

$$S_1 \not\leq N_1. \quad (3)$$

Next consider the case when $N_1 \neq 1$, i.e., $1 \neq Z = S_1 \cap N_1$. Let overbars denote passage to G/N_1 . Note that by definition of N_1 , $\mathcal{O}_{p'}(\overline{G}) = \overline{1}$. Since Z is Sylow in N_1 we have

$$\overline{S_1} \times \overline{S_2} = \overline{S_1} \times \overline{S_2} \cong (S_1/Z) \times S_2,$$

which is a direct product of strongly closed subgroups of \overline{G} . By induction and (1) and (2) we therefore have

$$\overline{G} = \overline{K_1} \times \overline{K_2}. \quad (4)$$

Let G_2 be the complete preimage of K_2 in G , so that $G_2 < G$ by (3) and (4). As $G_2 \trianglelefteq G$ we have $\mathcal{O}_{p'}(G_2) = 1$. By induction applied to $Z \times S_2$ in G_2 and by (1) we obtain

$$\langle Z^{G_2}, S_2^{G_2} \rangle = \langle Z^{G_2} \rangle \times K_2. \quad (5)$$

Now (4) shows that $K_1 \cap K_2 \leq N_1 \cap K_2$ and (5) implies the latter intersection has order prime to p , hence we are done in this case. This argument proves

$$\mathcal{O}_{S_1}(G) = 1.$$

We now apply the classifications in [FlFo, Theorems 1.1, 1.2] to S_1 to obtain

$$E(K_1) = F^*(K_1) = L_1 \times L_2 \times \cdots \times L_r \quad (6)$$

where each L_i is a simple group of known type and $S_1 \cap L_i$ is a nontrivial strongly closed subgroup of L_i but not Sylow in L_i , for $1 \leq i \leq r$. More precisely, S_1 always contains $\Omega_1(Z(P_i))$ where P is a Sylow p -subgroup of G containing $S_1 \times S_2$ and $P_i = P \cap L_i$ is a Sylow p -subgroup of L_i . Since S_2 centralizes S_1 , it normalizes each L_i ; hence $S_2 \leq \cap_{i=1}^r N_G(L_i) \trianglelefteq G$. Consequently

$$K_2 \text{ normalizes } L_i \quad \text{for } 1 \leq i \leq r.$$

Since S_2 is strongly closed and normalizes L_i , by Lemma 2.3(2) in [FlFo] either S_2 centralizes L_i or $S_2 \cap L_i \neq 1$. In the latter situation, however, $S_2 \cap Z(P_i) \neq 1$, contrary to $\Omega_1(Z(P_i)) \leq S_1$. This forces

$$S_2 \text{ centralizes } L_i \quad \text{for } 1 \leq i \leq r.$$

By (6) this means S_2 centralizes $F^*(K_1)$. Thus K_2 centralizes $F^*(K_1)$, and so $K_1 \cap K_2 \leq C_{K_1}(F^*(K_1)) = 1$, as needed to complete the proof.

References

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- [Go] D. Goldschmidt, Strongly closed 2-subgroups of finite groups, *Ann. Math.* 102(1975), 475–489.
- [Gr] D. Gorenstein, *Finite Groups*, AMS Chelsea, Providence RI, 1980.