

HOMOGENEOUSLY GENERATED GROUPS

RICHARD FOOTE AND LENNY JONES

Department of Mathematics and Statistics, University of Vermont
Burlington VT 05405, and
Department of Mathematics, Shippensburg University, Shippensburg, PA 17257
emails: lkjone@ship.edu and foote@math.uvm.edu

ABSTRACT. We develop basic properties of groups that contain exactly k elements of order n and are generated by these elements (called (k, n) -homogeneous groups). We then give a complete classification of (n, n) -homogeneous groups.

1. Introduction

This paper evolved out of our own study, and that of our students, of groups containing exactly n elements of order n . Minimal groups with this property are generated by their elements of order n , and we call these *n -homogeneous groups*. Our initial investigations culminated in the paper [1], which gives an elementary classification of abelian n -homogeneous groups. In this paper we complete the picture by classifying all n -homogeneous groups. Before doing so, we establish some basic properties of a larger class of finite groups—ones whose generators all have the same order—and see that this study leads to a potentially interesting family of irregular p -groups.

For positive integers k and n we say a group is a *(k, n) -homogeneous group* if it contains exactly k elements of order n and is generated by these elements. An n -homogeneous group is thus an (n, n) -homogeneous group. Finally, we say a group is *homogeneously generated* if it is a (k, n) -homogeneous group for some k, n .

If a group G contains an element of order n , then the set of all elements of G of order n generates a characteristic subgroup, G_n , of G on which G acts as automorphisms. Let $K = C_G(G_n)$ be the kernel of this action. If G is finite, then G has k elements of order n for some k , i.e., G_n is a (k, n) -homogeneous group, and the structure of G/K is thereby restricted by the isomorphism type of G_n . In Section 2 we show that (k, n) -homogeneous groups are necessarily finite, and we give an explicit bound for their orders as a function of k and n . We also show that many families of finite groups are homogeneously generated, and that the minimal non-homogeneously generated groups are irregular p -groups.

Note that if a group G contains exactly k elements of order n but is not necessarily generated by them, then so too does, for example, $G \times H$ for any subgroup H of order prime to n . Thus the hypothesis that the elements of order n generate G provides an appropriate restriction for focusing on the essential groups in the context of all groups containing k elements of order n .

In Section 3 we establish the main result of the paper, which is given in Theorem 1: the classification of n -homogeneous groups. In order to state this theorem we list notation for some families of finite groups. For a prime p and a positive integer $k \geq 2$, let

$$M_{k+1}(p) = \langle x, y \mid x^{p^k} = y^p = 1, y^{-1}xy = x^{1+p^{k-1}} \rangle$$

(sometimes called the *modular p-group* of order p^{k+1}). For a positive integer $a \geq 1$ define

$$H_a = \langle x, y \mid x^{2^a} = y^3 = 1, x^{-1}yx = y^{-1} \rangle.$$

Let $GL_2^*(3)$ denote the group of order 48 containing $SL_2(3)$ as a subgroup of index 2 and with generalized quaternion Sylow 2-subgroups. The remainder of our notation is standard, as found for example in [2].

Theorem 1. Let $n > 1$ be an integer. A group G is n -homogeneous if and only if $n = 2^a 3^b$ and one of the following holds:

1. G is abelian and isomorphic to one of the following groups:

- (i) $Z_{2^a} \times Z_2$, $a \geq 2$, $b = 0$
- (ii) $Z_2 \times Z_2 \times Z_{3^b}$, $a = 1$, $b \geq 1$
- (iii) $Z_{2^a} \times Z_{3^b} \times Z_3$, $a \geq 1$, $b \geq 2$, or

2. G is non-abelian and isomorphic to one of the following groups:

- (i) $M_{a+1}(2)$, $a \geq 3$, $b = 0$
- (ii) $Z_{2^a} \times M_{b+1}(3)$, $a \geq 1$, $b \geq 2$
- (iii) $H_a \times Z_{3^b}$, $a \geq 1$, $b \geq 1$
- (iv) $Q_8 \times Z_{3^b}$, $a = 2$, $b \geq 1$
- (v) $S_4 \times Z_{3^b}$, $a = 2$, $b \geq 1$
- (vi) $GL_2(3) \times Z_{3^b}$, $a = 3$, $b \geq 1$
- (vii) $GL_2^*(3) \times Z_{3^b}$, $a = 3$, $b \geq 1$.

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2. Results on Homogeneously Generated Groups

Throughout this section k and n are integers with $n > 1$.

Proposition 1. Every homogeneously generated group is finite.

Proof: Let G be a (k, n) -homogeneous group. Since each element of order n has at most k conjugates, its centralizer has finite index in G . Thus the intersection of the centralizers of all elements of order n has finite index. Since this intersection centralizes a generating set for G , it is contained in the center of G , that is, $Z(G)$ is of finite index in G . By a result of Schur, the commutator subgroup, G' , is finite. The abelian group G/G' is generated by a finite collection of elements of orders dividing n (the cosets of the elements of order n in G), hence is a finite group, as desired.

The method of proof above also leads to an upper bound on the order of a (k, n) -homogeneous group: If G is (k, n) -homogeneous, then the centralizer of each element of order n has index at most k . The intersection of these centralizers has index at most $k!$, that is

$$|G : Z(G)| \leq k!.$$

Let $r = |G : Z(G)|$. Then by [3] every element in G' can be written as a product of $d(r)$ commutators, where $d(r)$ is the number of divisors of r . There are at most $\binom{r-1}{2}$ nontrivial commutators, so

$$|G'| \leq \binom{r-1}{2}^{d(r)}. \quad (2.1)$$

Now each element of G of order n generates a cyclic group containing $\phi(n)$ elements of order n . Thus G has exactly $k' = k/\phi(n)$ cyclic subgroups of order n ; and so G is generated by k' elements, one generator selected from each of these cyclic groups. Thus the homomorphic image G/G' is generated by k' elements, each of which has order dividing n . Since G/G' is an abelian group of exponent dividing n and rank at most k' , we have

$$|G/G'| \leq n^{k/\phi(n)}. \quad (2.2)$$

Combining these bounds gives:

Proposition 2. The order of a (k, n) -homogeneous group is at most $\binom{r-1}{2}^{d(r)} n^{k/\phi(n)}$, where $r = k!$ and $d(r)$ is the number of divisors of r . In particular, for fixed k and n there are only a finite number of non-isomorphic (k, n) -homogeneous groups.

The above bound is quite crude—tighter bounds can be given, in particular, in (2.1).

We next give examples of homogeneously generated groups.

Proposition 3. A finite group whose composition factors are all non-abelian is generated by a single conjugacy class. In particular, such groups are homogeneously generated.

Proof: Let G be a finite group whose composition factors are all non-abelian and let M be a minimal normal subgroup of G . By induction, $\overline{G} = G/M$ is generated by a single conjugacy class. Let x be an element of G such that xM is a representative of this class, and let G_0 be the normal subgroup of G generated by the conjugates of x . If $M \leq G_0$, then $G_0 = G$ and the conclusion holds. If $M \not\leq G_0$, then $M \cap G_0 = 1$ and $G = M \times G_0$. In the latter case let m be any nonidentity element of M and replace x by xm . Since M is a direct product of non-abelian simple groups, it is not centralized by xm , and since $xmM = xM$, it follows by our arguments that G is now generated by the conjugates of xm , as desired.

Note that by using the Feit-Thompson Theorem, this proof can be refined to show that the groups in Proposition 3 may be homogeneously generated by classes of 2-elements.

We say a finite group G is a *minimal non-homogeneously generated group* if G cannot be generated by elements of the same order but every proper subgroup is homogeneously generated (where the orders of generators may vary, depending on the subgroup).

Proposition 4. A minimal non-homogeneously generated group is an irregular p -group for some prime p . In particular, finite abelian groups are all homogeneously generated, as are p -groups of class at most p .

Proof: Let G be a minimal non-homogeneously generated finite group and let M be a maximal subgroup of G . By hypothesis M is generated by elements of order n . The conjugates of M are then also generated by elements of order n . Any two distinct conjugates of a maximal subgroup generate the whole group. Since G is not homogeneously generated, M must be normal. Since every maximal subgroup of G is normal, G is nilpotent, and hence is the direct product of its Sylow subgroups. Since clearly the direct product of homogeneously generated groups of relatively

prime orders is again homogeneously generated (by products of the generators), the minimality of G forces it to be a p -group for some prime p . Since G is not cyclic we may choose distinct maximal subgroups M_1, M_2 and M_3 of G with M_i generated by elements of order $n_i, i = 1, 2, 3$. Note that any pair of these subgroups generates G so the n_i are all distinct. After possibly relabeling, we may assume $n_1 < n_2 < n_3$. If G were regular, the order of any element of G would be bounded by the maximum order of elements in any set of generators of G . This would force $n_3 \leq n_2$, a contradiction. This completes the proof.

Note that the quasidihedral groups, $QD_{2^{m+1}}, m \geq 3$ are examples of minimal non-homogeneously generated groups since their maximal subgroups are of type Z_{2^m}, Q_{2^m} and D_{2^m} , no pair of which can be generated by elements of the same order. The authors' efforts to construct examples of minimal non-homogeneously generated p -groups for odd primes p —the smallest can be shown to have order at least p^{p+2} and exponent at least p^{p+1} —or to prove the nonexistence of such groups have not yet succeeded (we tend to believe that examples do exist).

3. Preliminary Results on n -homogeneous Groups

Throughout this section $n > 1$ is an integer and G is an n -homogeneous group. By Proposition 1, G is finite. As observed earlier, the cyclic subgroups of G partition the set of elements of order n , and each such cyclic subgroup contains $\phi(n)$ elements of order n (its generators). Thus the number of elements of order n is $r\phi(n)$, where r is the number of cyclic subgroups of order n ; in particular

$$\phi(n) \mid n \quad \text{and} \quad G \text{ has exactly } \frac{n}{\phi(n)} \text{ cyclic subgroups of order } n. \quad (3.1)$$

Write $n = \prod_{i=1}^m p_i^{a_i}$, where the p_i are distinct primes. Since

$$\frac{n}{\phi(n)} = \prod_{i=1}^m \frac{p_i}{p_i - 1} \in \mathbb{Z}$$

it follows by comparing powers of 2 in the numerator and denominator of the product that $m = 1$ or 2. If $m = 1$, then $p_1 = 2$, n is a power of 2, and G has exactly two subgroups of order n . If $m = 2$, then $n = 2^a 3^b$ with $a, b > 0$, and G has exactly three subgroups of order n . Note that in each case G permutes the subgroups of order n ; let K be the kernel of this action, i.e., the intersection of the normalizers of the cyclic subgroups of order n . Thus G/K is isomorphic to a subgroup of S_2 or S_3 respectively. Since K acts on each cyclic subgroup of order n , K' centralizes each of these. Since these generate G , $K' \leq Z(G)$, and, in particular, G is a solvable group. Furthermore, these arguments show that any element of order prime to 6 lies in the center of G . Thus the Sylow p -subgroups for $p \geq 5$ are direct factors of G , so since G is generated by $\{2, 3\}$ -elements, it must be a $\{2, 3\}$ -group. Indeed, when $m = 1$ this same reasoning shows G is a 2-group. We summarize these results as:

Theorem 2. If G is n -homogeneous for some integer $n > 1$, then one of the following holds:

1. $n = 2^a$ and G is a 2-group that has exactly two subgroups of order n , or
2. $n = 2^a 3^b$ with $a, b > 0$ and G is a $\{2, 3\}$ -group that has exactly three subgroups of order n .

Before we begin the proof of Theorem 1 we need some lemmas. The first two of these are well known, and can be found in [4, Section III.8] and [2, Section 5.4].

Lemma 1. Let p be a prime and assume that the p -group P has a unique cyclic subgroup of order p^k , for some $k \geq 1$.

1. If $p = 2$, then P is either cyclic, dihedral, generalized quaternion, or quasidihedral.
2. If p is odd, then P is cyclic.

Lemma 2. Let p be a prime and assume that the p -group P has a cyclic subgroup of index p .

1. If $p = 2$, then P is either cyclic, dihedral, generalized quaternion, quasidihedral, modular, or of type $Z_{2^k} \times Z_2$.
2. If p is odd, then P is cyclic, modular or of type $Z_{p^k} \times Z_p$.

Lemma 3. Assume the 2-group P has exactly two cyclic subgroups of order 2^k , for some $k \geq 1$, and that P is generated by these two subgroups. Then one of the following holds:

1. $k = 2$ and $P \cong Z_2 \times Z_4$, or
2. $k \geq 3$ and P is isomorphic to either $Z_{2^k} \times Z_2$ or $M_{k+1}(2)$.

Proof: Let these two cyclic subgroups of P be A and B . Since A permutes these two subgroups but normalizes itself, A normalizes B . Thus A , and likewise B , are normal in $\langle A, B \rangle = P$. Since P acts on the cyclic group A , P' centralizes A . Likewise P' centralizes B , so $P' \leq Z(P)$. Thus P has nilpotence class at most 2. It follows by inspection of the groups in Lemma 1(1) that P has an element x of order 2 not contained in A . Let $P_0 = \langle A, x \rangle$. Since P_0 has a cyclic subgroup of index 2 and has at least two elements of order 2, by Lemma 2, P_0 is either abelian of type $Z_{2^k} \times Z_2$, modular, dihedral, or quasidihedral. In the first two cases, P_0 itself contains two cyclic subgroups of order 2^k , so $P_0 = P$ and the desired conclusions hold. In the latter two cases, since P has nilpotence class at most 2, we must have $P_0 \cong D_8$ and $k = 2$. But now P is non-abelian, $P = AB$, and $[A, B] \leq A \cap B$. It follows that $|P| = 8$, and so $P = P_0$, a contradiction since D_8 does not satisfy the hypotheses of the lemma. This completes the proof.

We now immediately obtain:

Corollary 1. In the case when conclusion 1 of Theorem 2 holds, Theorem 1 is true.

The remainder of the paper now focuses on when conclusion 2 of Theorem 2 holds.

Lemma 4. Assume the 3-group P has exactly three cyclic subgroups of order 3^k , for some $k \geq 1$, and that P is generated by these three subgroups. Then $k \geq 2$ and P is isomorphic to either $Z_{3^k} \times Z_3$ or $M_{k+1}(3)$.

Proof: The proof of this result is similar but easier to establish than the preceding lemma.

Lemma 5. Assume the 2-group P has exactly three cyclic subgroups of order 2^k for some $k \geq 1$, and that P is generated by these three subgroups. Then one of the following holds:

1. $k = 1$ and $P \cong Z_2 \times Z_2$, or
2. $k = 2$ and $P \cong Q_8$.

Proof: Let A , B , and C be the three cyclic subgroups of order 2^k in P . Since the 2-group P permutes these by conjugation, exactly one or all three of them are normal. Furthermore, if P has a unique subgroup of order 2, then by Lemma 1, $P \cong Q_8$ and the lemma is proven. We assume therefore that P has more than one element of order 2.

Case I: A , B and C are normal in P .

In this situation the commutator of any two of A , B and C is contained in the intersection of the two subgroups. In particular, any pair that intersect trivially necessarily commute, hence generate their direct product. If $k = 1$, it follows that $P \cong Z_2 \times Z_2$. We therefore assume that $k \geq 2$.

Since $Z_{2^k} \times Z_{2^k}$ contains more than three cyclic subgroups of order 2^k , we must have that any pair of A , B and C have a nontrivial intersection. Thus A , B and C all contain the same element of order 2.

If $k = 2$ and P is abelian, then from the preceding paragraph P has order 8 or 16, and is of type $Z_4 \times Z_2$, $Z_4 \times Z_4$ or $Z_4 \times Z_2 \times Z_2$. None of these groups, however, has exactly three cyclic subgroups of order 4, a contradiction. If $k = 2$ and P is nonabelian, then choose notation so that A and B do not commute. Since $A \cap B \neq 1$, it follows that $AB = \langle A, B \rangle \cong Q_8$. Since Q_8 already contains three cyclic subgroups of order 4, we have that $P = AB \cong Q_8$, as asserted. It remains to consider when $k \geq 3$.

By Lemma 1 and a preceding observation, we can select from P an element x of order 2 not contained in A , B or C . Let $P_0 = \langle A, x \rangle$. Arguing as in the proof of Lemma 3, it follows that P has nilpotence class at most 2, and since $k \geq 3$, we have that P_0 is either $Z_{2^k} \times Z_2$ or modular. Likewise $P_1 = \langle B, x \rangle$ and $P_2 = \langle C, x \rangle$ are also of these isomorphism types. Note that each of P_0 , P_1 and P_2 contains exactly two of the three cyclic subgroups of order 2^k , and so each pair must share a common cyclic subgroup of order 2^k . Since x , together with any cyclic subgroup of order 2^k in P_i , generates P_i , it follows that $P_0 = P_1 = P_2$. But then $P = P_0$, contrary to P having three cyclic subgroups of order 2^k . This contradiction completes Case I.

Case II: A is normal in P but neither B nor C are normal in P .

In this case B and C are conjugate in P and $N = N_P(B)$ has index 2. Since $N \trianglelefteq P$, we also have that $N = N_P(C)$, and so $B, C \leq N$. In particular, B and C normalize each other, and A is not contained in N . A generator for A conjugates B into C , and so

$$AB = AC = \langle A, B, C \rangle = P. \quad (3.2)$$

Since N has exactly two subgroups of order 2^k , by Lemma 3, $P_0 = \langle B, C \rangle$ is either $Z_{2^k} \times Z_2$ or modular. In either case, P_0 has a unique subgroup W of type $Z_2 \times Z_2$. Since $P_0 \trianglelefteq P$ and W is characteristic in P_0 , we have that $W \trianglelefteq P$. Let $\langle z \rangle = B \cap C$. Since A has a unique element of order 2 and $W \trianglelefteq P$, there exists $x \notin A \cup B \cup C$ such that $W = \langle z, x \rangle$ and $[x, P] \leq \langle z \rangle$.

As we have argued before, $P_1 = \langle A, x \rangle$ is either dihedral, quasidihedral, modular or $Z_{2^k} \times Z_2$. In the latter two cases, P_1 contains two cyclic subgroups of order 2^k , and hence P_1 contains either B or C as well as A . By (3.2), $P_1 = P$. This is impossible because P would be generated by B or C together with x , and so P would be contained in N , a contradiction. This proves P_1 must be dihedral or quasidihedral.

Since $[x, A] \leq [x, P]$ has order at most 2, we must have that P_1 has nilpotence class at most 2, and so $P_1 \cong D_8$ and $k = 2$. Because A is not contained in $Z(P)$, it follows that AB is non-abelian. Also, $|AB : A| \neq 2$ since A does not normalize B . Thus AB must have order 16, $A \cap B = 1$, and $P = AB$. We argue that P has more than three cyclic subgroups of order 4: Namely, if $A = \langle u \rangle$ and $B = \langle w \rangle$, then $C = \langle w^u \rangle$. Then A , B , C and $\langle uw^2 \rangle$ are all distinct cyclic subgroups of order 4, a contradiction. This completes the proof of the lemma.

4. The Proof of Theorem 1

By Corollary 1 we are reduced to the case when $n = 2^a 3^b$ for some $a, b > 0$, where G is a $\{2, 3\}$ -group containing exactly three subgroups of order n . We treat the nilpotent case first.

Proposition 5. A nilpotent n -homogeneous group satisfies the conclusions to Theorem 1.

Proof: We have $G = T \times Q$ where T is the Sylow 2-subgroup and Q is the Sylow 3-subgroup of G . Moreover, T is generated by the cyclic subgroups of order 2^a and Q is generated by the cyclic subgroups of order 3^b . The number of cyclic subgroups of order n is therefore equal to the number of cyclic subgroups of T of order 2^a times the number of cyclic subgroups of Q of order 3^b . Since $b > 0$, exactly one of the following holds:

1. G has one cyclic subgroup of order 2^a and three cyclic subgroups of order 3^b , or
2. G has three cyclic subgroups of order 2^a and one cyclic subgroup of order 3^b .

Since the Sylow subgroups are generated by these cyclic subgroups, Lemmas 4 and 5 establish the proposition in both of these cases. This completes the proof.

We now complete the proof of Theorem 1. We may assume that G is not nilpotent and denote the three cyclic subgroups of order n by A , B , and C . Either all are normal, exactly one is normal, or none is normal in G . If all are normal, then again G is generated by normal nilpotent subgroups, implying that G is nilpotent, a contradiction.

Consider the case when exactly one is normal, choosing notation so that $A \trianglelefteq G$ but B and C are not normal. As in the proof of Lemma 5, B and C are conjugate and $N_G(B) = N_G(C)$ has index 2 in G . Hence B and C normalize each other, and so they generate a nilpotent group BC . But then G is generated by the normal nilpotent subgroups A and BC , and consequently G is nilpotent. This contradiction proves that

none of A , B or C is normal and so all are conjugate in G .

Let M be the kernel of the transitive action of G on $\{A, B, C\}$. Since each of A , B and C normalizes itself and interchanges the remaining two subgroups, $G/M \cong S_3$, and AM/M , BM/M and CM/M are the three Sylow 2-subgroups of G/M . Also, M acts on each cyclic group A , B , and C , and so M' centralizes them. Thus $M' \leq Z(G)$, and so M is nilpotent of class at most 2.

We next unravel the structure of the subgroup AM . Note that A is the unique cyclic subgroup of AM of order n because the other such subgroups of G lie in different cosets mod M . Thus

$$A \trianglelefteq AM \quad \text{and} \quad AM \text{ is nilpotent.}$$

Now AM has a unique cyclic subgroup of order n , hence the Sylow 2-subgroup and Sylow 3-subgroup of AM have unique cyclic subgroups of order 2^a and 3^b respectively. By Lemma 1, the Sylow 2-subgroup of AM is cyclic, dihedral, quasidihedral or generalized quaternion, and the Sylow 3-subgroup of AM is cyclic (and central in AM). Note that AM contains a Sylow 2-subgroup of G , so these have the same specified isomorphism type too. By symmetry, the same conclusions hold for BM and CM .

Let Y be the Sylow 3-subgroup of M and let Y be of index 3 in a Sylow 3-subgroup Y^* of G . Since Y commutes with A , B and C , we have that $Y \leq Z(G)$. Since Y^*/Y is cyclic, Y^* is abelian. A generator for A inverts the Sylow 3-subgroup Y^*M/M of G/M but centralizes Y . Thus Y^* cannot be cyclic and hence $Y^* = Y_0 \times Y$, for some subgroup Y_0 of Y of order 3. By Gaschütz's

Theorem, since the central subgroup Y is a direct factor of a Sylow 3-subgroup, Y splits off of G as well:

$$G = G_0 \times Y, \quad \text{for some subgroup } G_0 \text{ of } G.$$

Consider first when G_0 , (or equivalently G) has a normal 2-complement, K . In this situation $K = Y^*$ and $\overline{G} = G/K$ is generated by cyclic groups of 2-power order. Moreover, since A , B and C are conjugate by elements of Y^* , we have $\overline{A} = \overline{B} = \overline{C}$, and so \overline{G} is cyclic. In other words, a Sylow 2-subgroup of G is cyclic of order 2^a . It follows easily that $G \cong H_a \times Z_{3^b}$, and so Theorem 1 is verified in this case.

Finally, suppose G_0 does not have a normal 2-complement. Note that G_0 is solvable of order $2^c 3$ with Sylow 2-subgroups that are cyclic, dihedral, quasidihedral or generalized quaternion. By the basic theory of solvable groups (see [2, Section 6.3]), G_0 acts as automorphisms of $O_2(G_0)$ with the kernel of this action the center of $O_2(G_0)$. It is easily seen that one of the following holds:

1. $O_2(G_0)$ is a fourgroup, $G_0/O_2(G_0) \cong S_3$ and $G_0 \cong S_4$,
2. $O_2(G_0) \cong Q_8$, $G_0/O_2(G_0) \cong S_3$, Sylow 2-subgroups of G_0 are quasidihedral of order 16 and $G_0 \cong GL_2(3)$, or
3. $O_2(G_0) \cong Q_8$, $G_0/O_2(G_0) \cong S_3$, Sylow 2-subgroups of G_0 are generalized quaternion of order 16 and $G_0 \cong GL_2(3)^*$.

In all three cases Theorem 1 holds.

Conversely, it is straightforward to check that each group on the list in the statement of Theorem 1 is n -homogeneous for the specified n . This completes the proof of Theorem 1.

References

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