A CHARACTERIZATION OF FINITE GROUPS CONTAINING A STRONGLY CLOSED 2-SUBGROUP

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1. Introduction

Throughout this paper G is a finite group and S is a subgroup of G. When S is contained in the subgroup H of G, we say S is *strongly closed in H with respect to G* if whenever $s \in S$ and $g \in G$ are such that $s^g \in H$, then $s^g \in S$. In other words, every G-conjugacy class of elements of S intersected with H is contained in S. In the case where S is a p-group for some prime p we say that S is *strongly closed* if it is strongly closed in some Sylow p-subgroup containing it. One easily sees that S is strongly closed if and only if it is strongly closed in $N_G(S)$ with respect to G, so strong closure is independent of the choice of Sylow p-subgroup containing S.

Results about strongly closed subgroups, especially 2-subgroups, have played a pivotal role in the Classification of Finite Simple Groups. Glauberman's Z^*-Theorem, which determined the normal closure of a strongly closed subgroup of order 2, formed the cornerstone of a succession of results which were used extensively in 2-local analysis. One milestone was Goldschmidt's generalization of the Z^*-Theorem, [9], which determined the normal closure of groups with an abelian strongly closed 2-subgroup. In [16], Rowley extended Goldschmidt's analysis to characterize groups with a strongly closed 2-subgroup of nilpotence class 2. Goldschmidt proved further powerful theorems on strongly closed 2-subgroups in [10]; in particular, his results on products of strongly closed 2-subgroups greatly simplified some earlier classification proofs in which product fusion occurred. The purpose of this paper is to complete the picture by characterizing exactly when a finite group possesses a strongly closed 2-subgroup.

An obvious example of a strongly closed p-subgroup of G is when S is a Sylow p-subgroup of some normal subgroup of G. Note that this happens if and only if S is a Sylow p-subgroup of its normal closure $\langle\, S^G \,\rangle$ in G. This paper determines for $p = 2$ the composition factors of any group G possessing a strongly closed 2-subgroup which does *not* arise this way, i.e., which is not a Sylow 2-subgroup of its normal closure.

The nonidentity p-subgroup S_0 is a *minimal strongly closed subgroup of* G if S_0 is strongly closed but no nontrivial proper subgroup of S_0 is also strongly closed. The main result of this paper is:

Theorem 1. *Let* S_0 *be a minimal strongly closed 2-subgroup of the finite group* G. *Then either* S_0 *is elementary abelian or* S_0 *is a Sylow 2-subgroup of the normal subgroup* $\langle S_0^G \rangle$.

By inspection of Goldschmidt's list of groups possessing a strongly closed abelian 2-subgroup we obtain:

Corollary 1. *Let* S_0 *be a minimal strongly closed 2-subgroup of* G *and let* $G_0 = \langle S_0^G \rangle$. *If* S_0 *is not a Sylow 2-subgroup of* G_0, *then* $G_0/O(G_0)$ *is isomorphic to the direct product of* k *copies of either* $U_3(2^n)$ *or* $Sz(2^n)$, *for some* $n \geq 2$ *(permuted transitively by* G *under conjugation), and* S_0 *is elementary abelian of order* 2^{kn}.

Since the property of strong closure carries over to quotient groups (cf. Lemma 1), Corollary 1 can be applied to restrict the structure of groups possessing an arbitrary strongly closed 2-subgroup as follows. Let R be any 2-subgroup of a finite group G. If N_1 and N_2 are normal subgroups of G with $R \cap N_i \in Syl_2(N_i)$ for both $i = 1, 2$, then $R \cap N_1N_2$ is a Sylow 2-subgroup of N_1N_2. Thus *there is a unique largest normal subgroup* N *of* G *for which* $R \cap N \in Syl_2(N)$; *denote this subgroup by* $\mathcal{O}_R(G)$. Thus

$$R \text{ is a Sylow 2-subgroup of } \langle R^G \rangle \text{ if and only if } R \leq \mathcal{O}_R(G).$$

Note that $O(G/\mathcal{O}_R(G)) = 1$; in particular, if $R = 1$ is the identity subgroup, $\mathcal{O}_1(G) = O(G)$. Also, $R\mathcal{O}_R(G)/\mathcal{O}_R(G)$ does not contain the Sylow 2-subgroup of any nontrivial normal subgroup of $G/\mathcal{O}_R(G)$.

In this notation Theorem 1 and Lemma 1 easily give the following results about arbitrary strongly closed 2-subgroups:

Theorem 2. *Let* G *be a finite group which has a strongly closed 2-subgroup* S. *Assume* S *is not a Sylow 2-subgroup of* $\langle S^G \rangle$, *and let* $\overline{G} = G/\mathcal{O}_S(G)$. *Then* $\overline{S} \neq 1$ *and* $\langle \overline{S}^{\overline{G}} \rangle$ *is isomorphic to the direct product of groups isomorphic to* $U_3(2^n)$ *or* $Sz(2^n)$ *for various* $n \geq 2$ *with* $\overline{S}$ *being the center of a Sylow 2-subgroup of this direct product. In particular,* $\overline{S}$ *is elementary abelian.*

Conversely, observe that any finite group which has a composition factor of type $U_3(2^n)$ or $Sz(2^n)$, for some $n \geq 2$ is seen to possess a strongly closed 2-subgroup that is not a Sylow 2-subgroup of its normal closure in G.

Theorem 3. *Let* G *be a finite group with no composition factors isomorphic to* $U_3(2^n)$ *or* $Sz(2^n)$, *for all* $n \geq 2$. *A 2-subgroup* R *of* G *is a Sylow 2-subgroup of some normal subgroup of* G *if and only if* R *is strongly closed in* G.

The proof of Theorem 1 relies on the Classification of Finite Simple Groups. One quickly reduces to the case where a minimal counterexample, G, is a simple group having a nonabelian strongly closed 2-subgroup S which is properly contained in a Sylow 2-subgroup T of G. The remainder of the proof involves investigation of the families of simple groups to determine that this does not happen. It is convenient to quote Goldschmidt's classification, but Rowley's intricate analysis of the class-two situation is not used.

In general terms the proof divides naturally into separate analyses of the characteristic 2-type groups and the component type groups. In many of the Chevalley groups over fields of characteristic 2, $P = N_G(Z(T))$ is a maximal parabolic subgroup of G and $P = QLH$, where $Q = O_2(P)$, H is an odd order Cartan subgroup, and L is the quasisimple component of a Levi factor. Frequently L is generated by G-conjugates of $Z(T)$ and LH acts irreducibly on Q/Q'. For such groups, since $S \cap Z(T) \neq 1$, it follows immediately that $\langle S^P \rangle$ contains a Sylow 2-subgroup of G. Hence by induction — provided L is not $U_3(2^n)$ or $Sz(2^n)$ — we obtain $S = T$, a contradiction. In the analysis in groups of component type it is shown at the outset that S contains a conjugate of a Sylow 2-subgroup of every component of the centralizer of each involution in G. This result together with some knowledge of the action of G on its class of fundamental $SL_2(q)$ subgroups again shows that S contains subgroups which generate the Sylow 2-subgroup T. In both the even and odd characteristic type groups the bulk of the work consists of eliminating certain groups of low rank. Some subtleties occur in groups which do possess strongly involution closed 2-subgroups.

One might hope, in the best case, to find a proof of Theorem 1 which is independent of the Classification or, as a next recourse, to have a proof which relies on a short list of "axiomatic" properties that most simple groups are easily seen to satisfy (e.g., ones that follow directly from properties of BN-pairs). The complexity of both [9] and [16] suggests the unlikelihood of an independent proof, even assuming G were a K-group. In both these papers the hypothesized low nilpotence class of S provides powerful inductive restrictions on the normal closure of S in 2-local subgroups containing it. In the general situation herein, however, essentially the only information available from the induction is that S is a Sylow 2-subgroup of $\langle S^H \rangle$ for any subgroup H containing S. Without some a priori knowledge of the structure of the maximal 2-locals containing T or their interplay, this condition appears to be too weak to appreciably restrict G. Even in the case of groups of component type where — assuming the B-conjecture — we can show that S contains a Sylow 2-subgroup of each component of $C_G(t)$, for all involutions $t \in T$, it is not clear in an unknown simple group G just how much of T these 2-subgroups generate.

The deduction of Theorem 2 from Theorem 1 is given in Section 3. Theorem 3 follows immediately from Theorem 2.

An application of Theorem 1 appears in [8].

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2. Proof of the Main Theorem

Lemma 1. *Let S be a strongly closed p-subgroup of G, for some prime p.*

1. *S is weakly closed in any Sylow p-subgroup containing it.*
2. *If N is a normal subgroup of G, then SN/N is strongly closed in G/N.*

Proof. The first assertion is immediate from the definition of a strongly closed subgroup. To show the second let $S \leq T \in Syl_p(G)$ and let $\overline{G} = G/N$, so that $\overline{T} \in Syl_p(\overline{G})$. Suppose $\overline{s} \in \overline{S}$ and $\overline{s}^{\overline{g}} \in \overline{T}$. We may pick a coset representative $s \in S$ for $\overline{s}$. Now $s^g \in SN$, and since $T \cap SN \in Syl_p(SN)$, there is some $n \in N$ such that $s^{gn} \in T$. Since S is strongly closed in T, $s^{gn} \in S$, and therefore $\overline{s}^{\overline{gn}} = \overline{s}^{\overline{g}} \in \overline{S}$, as desired.

Lemma 2. *If S is any strongly closed 2-subgroup of G and N is a core free S-invariant subgroup of G with $S \cap N = 1$, then S centralizes N.*

Proof. This is Lemma 2.4 of [10]. It follows easily by using the Z^*-Theorem and induction on $|S|$ in the group $G = SN$.

We begin the proof of Theorem 1. For the remainder of the paper G is assumed to be a counterexample of minimal order, so that G has a minimal strongly closed 2-subgroup S such that S is not a Sylow 2-subgroup of $\langle S^G \rangle$. Let $S \leq T$ for some Sylow 2-subgroup T of G. In this setting, for every proper subgroup H of G containing S, since some nontrivial subgroup S_0 of S is a minimal strongly closed subgroup with respect to H, the conclusion of Theorem 1 applies to $\langle S_0^H \rangle$. By induction, Theorems 2 and 3 apply in H as well.

We introduce some terminology that will help streamline the arguments, particularly in the latter stages of the proof of Theorem 1. For any subgroup H of G we say *H is an S-group if some (hence every) Sylow 2-subgroup of H lies in a G-conjugate of S.* Note that

1) subgroups and conjugates of S-groups are S-groups, and

2) if H and K are S-groups with H normalizing K, then HK is an S-group.

2.1. $O(G) = 1$.

This follows from the minimality of G together with the fact that the hypothesis carry over to $G/O(G)$, by Lemma 1.

2.2. $G = \langle S^G \rangle$.

Let $N = \langle S^G \rangle$ and assume by way of contradiction that $N \neq G$. Note that since G is a counterexample, $S \notin Syl_2(N)$, i.e., N is not an S-group. Since S is weakly closed in a Sylow 2-subgroup of N, Frattini's Argument gives that $G = N_G(S)N$. Let S_0 be a subgroup of S which is minimal strongly closed with respect to N, and let $M = \langle S_0^N \rangle$. The minimality of S with respect to G together with the Frattini factorization for G force

$$\langle S^G \rangle = \langle M^G \rangle = \langle M^{N_G(S)} \rangle. \tag{2.2.1}$$

By induction in N, either $S_0 \in Syl_2(M)$, or M is a direct product of simple groups of type $U_3(2^n)$ or $Sz(2^n)$. In the former case, it follows from (2.2.1) and the remarks preceding Theorem 2 that S is a Sylow 2-subgroup of $\langle M^G \rangle$, contrary to the choice of G. If, on the other hand, M is a direct product of the prescribed simple groups, so too is $\langle M^G \rangle$. Since these are permuted transitively by G and S_0 is the center of a Sylow 2-subgroup of M, again Theorem 1 is seen to hold in G, a contradiction.

2.3. *G is simple.*

Let $Q = O_2(G)$. If $S \cap Q \neq 1$, then since $S \cap Q$ is strongly closed in G, the minimality of S forces $S \leq Q$. Since S is weakly closed in T, we get that $S \trianglelefteq G$, contrary to G being a counterexample. Thus $S \cap Q = 1$.

Since $S \trianglelefteq T$, $[S, Q] \leq S \cap Q = 1$. Since $F^*(G) = E(G)Q$ and S centralizes Q, S must act nontrivially on $E(G)$. By Lemma 2 and the minimality of S we have $S \leq E(G)$. By 2.2, $G = E(G)$. Since S normalizes every component of G but does not centralize some component L, again Lemma 2 forces $S \cap L \neq 1$. Thus $S \leq L$, and again 2.2 gives that $L = G$ is quasisimple. It remains to show that $Z(G) = 1$, i.e., $Q = 1$.

Suppose $Q \neq 1$ and let bars denote passage to G/Q, so that $\overline{S} \cong S$. Since strong closure inherits to quotient groups, $\overline{S}$ is a minimal strongly closed subgroup of $\overline{G}$. By induction either $\overline{S}$ is a Sylow 2-subgroup of $\langle \overline{S}^{\overline{G}} \rangle$, or $\overline{G} \cong U_3(2^n)$ or $Sz(2^n)$ with S an elementary abelian group projecting onto $Z(\overline{T})$. In the former case, since a Sylow 2-subgroup of $\overline{G}$ splits over Q, by Gaschütz's Theorem G cannot be perfect, a contradiction. Assume therefore that the latter case holds. Since G has a 2-fold cover, by inspection of the Schur multipliers of the unitary and Suzuki groups (as summarized in [6]), G is a 2- or 2^2-fold cover of $Sz(8)$. Thus $SQ = S \times Q \cong E_{16}$ or E_{32} respectively. Let SQ_0 be a hyperplane of SQ and let H be a Cartan subgroup of G normalizing T (here $|H| = 7$). Since S is H-invariant (by strong closure) and Q_0 is a central subgroup, H acts on T/SQ_0 (which has order 16). Since H acts transitively on $(T/SQ)^\#$ and since $T' \leq SQ$, this action of H forces T/SQ_0 to be elementary abelian. Thus a complement to Q_0 in Q is not contained in the Frattini subgroup of T, and Gaschütz's Theorem again shows that G is not perfect. This contradiction completes the proof of 2.3.

2.4. *S is not abelian, and G is not isomorphic to $L_2(2^n)$, $U_3(2^n)$, $Sz(2^n)$, $Re(3^n)$, J_1, or $L_2(q)$, $q \equiv 3, 5 (mod\ 8)$, for any n.*

It is convenient to quote Goldschmidt's classification [9]: if S were abelian, then G would be one of the Goldschmidt groups listed in the conclusion of 2.4. This would be contrary to G being a counterexample, and so 2.4 must hold.

2.5. *T/S is not cyclic.*

If T/S were cyclic, then by the generalized Thompson Transfer Lemma (eg., (37.4) in [4]) an element of minimal order in $T - S$ would be conjugate in G to some element of S. This would violate the strong closure of S.

2.6. *If S contains a representative of each G-conjugacy class of involutions, then $T/\Omega_1(T)$ is not cyclic. In particular, if G has one class of involutions, then $T/\Omega_1(T)$ is not cyclic.*

This is immediate from 2.5.

2.7. *G does not have a subgroup M with the following properties:*

1. *M contains T,*
2. *$F^*(M) = O_2(M)$,*
3. *if $\overline{M} = M/O_2(M)$, then either $\overline{M} \cong A_5 \wr Z_2$ or $F^*(\overline{M})$ is a quasisimple subgroup of index at most 2 in $\overline{M}$,*
4. *$F^*(\overline{M})$ is not isomorphic to $U_3(2^n)$ or $Sz(2^n)$, for any $n \geq 2$,*
5. *$[O_2(M), F^*(\overline{M})] = O_2(M)$, and*
6. *if $Q = O_2(M)$ and $Z = \Omega_1(Z(T))$, then one of the following holds:*
   - *i. Q is abelian,*
   - *ii. Q is extraspecial, $Q = O_2(C_G(Z(Q)))$ and $\overline{M}$ acts irreducibly on Q/Q', or*
   - *iii. all involutions in Z are conjugate in G and $z^g \in T - Q$ for some $z \in Z$, $g \in G$.*

Assume G does possess such a subgroup and let Q and Z be as defined above. Hypotheses (3) and (5) imply that M' is a perfect subgroup of index at most 2 in M. Since M does not have any composition factors isomorphic to $U_3(2^n)$ or $Sz(2^n)$, by induction Theorem 3 gives that $S \in Syl_2(\langle S^M \rangle)$.

If $S \not\leq Q$, then by the structure of M hypothesized in (3) and (5), $M' \leq \langle S^M \rangle$. But then $|T : S| \leq 2$, contrary to 2.5. Thus

$$S \leq Q \quad \text{and} \quad S \trianglelefteq M. \tag{2.7.1}$$

By 2.4, Q is nonabelian, hence hypothesis (6.i) cannot hold. If (6.iii) held, then since $S \trianglelefteq T$, $Z \cap S \neq 1$. The strong closure of S would then force all G-conjugates of involutions in Z that belong to T to lie in S, and hence in Q, contrary to the assumption: $z^g \in T - Q$. Thus (6.iii) cannot hold either.

Finally, assume (6.ii) holds. In this case $Z = Z(Q) = Q' = \langle z \rangle$ has order 2. Since $Z < S$, the irreducible action of $\overline{M}$ on Q/Z together with (2.7.1) forces

$$S = Q.$$

By the Z^*-Theorem, $z^g \in Q - \langle z \rangle$ for some $g \in G$. Let $S_0 = C_Q(z^g)$. The action of M forces Q to have width at least 2, so S_0 is nonabelian. Because $Q^g \trianglelefteq C_G(z^g)$ we have that $S_0 Q^g$ is a 2-group. By strong closure therefore, $S_0 \leq Q^g$. But now $\langle z \rangle = S_0' = (Q^g)' = \langle z^g \rangle$, a contradiction. This completes the proof of 2.7.

2.8. *G is not isomorphic to A_n, for any $n \geq 5$.*

The groups A_5, A_6 and A_7 are eliminated by the second sentence of 2.6. Since A_8 contains a subgroup $M \cong E_8L_3(2)$ satisfying the hypotheses of 2.7, G is not isomorphic to A_8.

If n is odd, then A_{n-1} contains a Sylow 2-subgroup of G; it follows by inductively applying Theorem 3 in this subgroup that G is not a minimal counterexample. It therefore remains to consider when $n = 2m \geq 10$. Let

$$Q = \langle\, (1\,2)(3\,4),\ (3\,4)(5\,6), \ldots\ldots, (2m{-}3\;2m{-}2)(2m{-}1\;2m),\ (2m{-}1\;2m)(1\,2)\,\rangle$$

and let $M = N_{A_n}(Q) \cong 2^{m-1}\Sigma_m$. Then M satisfies the hypotheses of 2.7, and the remaining alternating groups are thereby eliminated.

2.9. *G is not isomorphic to a Chevalley group (untwisted or twisted) over a field of characteristic 2 (including the Tits simple group).*

Let $q = 2^n$, $n \geq 1$. An end-node maximal parabolic subgroup, P_1, for each of the Chevalley groups (untwisted or twisted) is described in detail in [7] and [13]. (For the classical groups these parabolics are the stabilizers of a totally isotropic one-dimensional space in the natural module.) For the groups of BN-rank 2 the other maximal parabolic, P_2, is also described in [13]. In each group $P_i = Q_iL_iH$, where $Q_i = O_2(P_i)$, L_i is the component of a Levi factor of P_i and H is an odd order Cartan subgroup. Except in $F_4(q)$ and some groups over $\mathbb{F}_2$ (which will be dealt with separately), for some $i \in \{1, 2\}$ the group $M = O^{2'}(P_i)$ satisfies hypotheses 1, 2, 3, 5 and 6 of 2.7. Basic information about this parabolic is listed in Table I. The last column of Table I indicates which of the three alternatives in hypothesis 6 of 2.7 holds. The verifications of the fusion hypotheses when 6.iii holds may be found in [7], [15], [17] and [18]. Except for the groups in the last four rows of the table, the families also satisfy hypothesis 4 of 2.7, and so these groups are eliminated. The groups in the last four lines of Table I, where some hypothesis of 2.7 does not hold, require some additional argument.

Groups over $\mathbb{F}_2$ among those listed in Table I where the Levi factor fails to be quasisimple (but have not already been dealt with) are: $L_3(2)$, $U_4(2)$, $S_6(2)$, $G_2(2)'$, $U_5(2)$ and ${}^2F_4(2)'$. Both $L_3(2)$ and $G_2(2)'$ are eliminated by 2.6. The groups $U_4(2)$ and $S_6(2)$ contain subgroups of types 2^4A_5 and $2^6L_3(2)$ respectively satisfying the hypotheses of 2.7, and are thus not possibilities for G either. The groups $U_5(2)$ and ${}^2F_4(2)'$ are eliminated below.

In $U_5(q)$ for $q \geq 2$ the unipotent radical of the parabolic P_1 is special of type q^{1+6} with $Z = Z(T) = Z(Q_1)$ and all involutions in Z conjugate in P_1 (so $Z \leq S$). As in the other unitary groups, for some $z \in Z$ and $g \in G$, $z^g \in T - Q_1$. Now $L_1 \cong U_3(q)$ acts irreducibly on Q_1/Z and, by the strong closure of S, $S \cap Q$ is normal in P_1. Since $z^g \in S$ and $[Q_1, z^g] \leq S \cap Q_1$, the irreducible action of L_1 forces $Q_1 \leq S$. But now there is a root group U of type $U_3(q)$ with U contained in Q_1 such that $T = Q_1U^x$, for some $x \in G$. Since $U \leq S$, this forces $S = T$, a contradiction.

Table I

Group	Parabolic	$\lvert Q\rvert$	$L/Z(L)$	2.7(6)
$L_k(q)$, $k \geq 3$	P_1	q^{k-1}	$L_{k-1}(q)$	i.
$S_{2k}(q)$, $k \geq 2$	P_1	q^{2k-1}	$S_{2k-2}(q)$	i.
$O_{2k}^{\pm}(q)$, $k \geq 4$	P_1	q^{2k-2}	$O_{2k-2}^{\pm}(q)$	i.
$U_k(q)$, $k \geq 4$, $k \neq 5$	P_1	q^{2k-3}	$U_{k-2}(q)$	iii.
$E_6(q)$	P_1	q^{21}	$L_6(q)$	iii.
$E_7(q)$	P_1	q^{33}	$O_{12}^{+}(q)$	iii.
$E_8(q)$	P_1	q^{57}	$E_7(q)$	iii.
${}^2E_6(q)$	P_1	q^{21}	$U_6(q)$	iii.
$G_2(q)$	P_2	q^5	$L_2(q)$	iii.
${}^3D_4(q)$	P_2	q^9	$L_2(q^3)$	iii.
$U_5(q)$	P_1	q^7	$U_3(q)$	
$F_4(q)$	P_1	q^{15}	$S_6(q)$	iii.
${}^2F_4(q)$, $q > 2$	P_1	q^{10}	$Sz(q)$	
${}^2F_4(2)'$	P_1	2^9	F_{20}	

Similarly, in ${}^2F_2(q)$ or ${}^2F_4(2)'$ the group $Z = Z(T)$ is elementary abelian of order q and all involutions in Z are conjugate in P_1. By [15] and [6], $z^g \in T - Q_1$ for some $z \in Z$ and $g \in G$. When $q = 2$, Q_1/Q_1' is an irreducible $\mathbb{F}_2F_{20}$-module of order 16; when $q > 2$, Q_1/Q_1' is an indecomposable extension of a one-dimensional trivial $\mathbb{F}_q$-module by a natural four-dimensional $\mathbb{F}_q$-module for $L_1 = Sz(q)$. In either case, arguing as in the preceding paragraph we obtain $Q_1 \leq S$. In the Tits simple group T/S is then cyclic, contrary to 2.5. When $q > 2$, by [15] there is a root group U of type $Sz(q)$ with U contained in Q_1 such that $T = Q_1U^x$, for some $x \in G$. Since $U \leq S$, this again forces $S = T$, a contradiction.

The structure of each parabolic subgroup in $F_4(q)$ is described in detail in [7]. In particular, the unipotent radical, Q, of the parabolic P_1 listed in Table I has the following decomposition: $Q = AB$ where $[A, B] = 1$, A is special of order q^9, B is elementary abelian of order q^7, and $A \cap B = Z(A)$ has order q. Moreover the Levi factor L acts irreducibly on $B/Z(A)$ and Q/B. Since S is nonabelian, $S \not\leq B$; and since $S \neq T$ it follows as usual that $S \leq Q$. The irreducible action of L on Q/B then forces S to cover Q/B, and so $S' = A' = Q'$. Now a graph automorphism interchanges the two end-node maximal parabolic subgroups, P and P_4, of $F_4(q)$ which contain T. The same reasoning applied in P_4 then shows that $S' = O_2(P_4)'$. This is absurd since P and P_4 do not normalize a nontrivial common 2-subgroup.

This completes the proof of 2.9.

2.10. *G is not isomorphic to any of the 26 sporadic simple groups.*

In Table II we list the 26 sporadic groups and a reason why each is eliminated as a possibility for G. Properties of the sporadic groups are neatly tabulated in [12] and the *Atlas*,

[6]. If G contains a simple subgroup of odd index that has already been eliminated, that subgroup is listed in the "Reason" column. When 2.7 is invoked, we list the subgroup M in the "Reason" column. If 2.7(6.ii) is used, one sees by inspection that the action of M on Q/Q' is irreducible; alternatively, by a result of Dempwolff and Wong [19], in the sporadic groups of $GF(2)$-type this action is reducible only in M_{24} and He (which are eliminated by applying 2.7 to subgroups other than the centralizer of a 2-central involution).

Only M_{12} and Ru require special consideration.

Table II

Group	Eliminated by	Reason
M_{11}	2.6	
M_{12}	see below	
M_{22}	2.6	
M_{23}	induction	contains M_{22}
M_{24}	2.7	2^4A_8
J_1	2.4	
J_2	2.7	$2^{1+4}A_5$
J_3	2.7	$2^{1+4}A_5$
J_4	2.7	$2^{11}M_{24}$
HS	2.7	$4^3L_3(2)$
He	2.7	$2^6 3{\cdot}\Sigma_6$
McL	2.6	
Suz	2.7	$2^{1+6}U_4(2)$
Ly	2.6	
Ru	see below	
$O'N$	2.7	$4^3L_3(2)$
Co_1	2.7	$2^{1+8}O_8^+(2)$
Co_2	2.7	$2^{10}M_{22}:2$
Co_3	2.7	2^4A_8
Fi_{22}	2.7	$2^{10}M_{22}$
Fi_{23}	2.7	$2^{11}M_{23}$
Fi'_{24}	2.7	$2^{11}M_{24}$
HN	2.7	$2^{1+8}(A_5 \wr Z_2)$
Th	2.7	$2^5L_5(2)$
B	2.7	$2^{1+22}Co_2$
M	2.7	$2^{1+24}Co_1$

If $G \cong M_{12}$ let H be a subgroup of G isomorphic to M_{11} with $T_0 = T \cap H \in Syl_2(H)$. Since $|T : T_0| = 4$ and S does not have order 4, $S \cap T_0 \neq 1$. Since $S \cap T_0$ is strongly closed in T_0 with respect to H, by induction $S \cap T_0 \in Syl_2(H)$, i.e., $T_0 \leq S$ and H is an S-group. But T_0 is not normal in T (as can be seen from the 3-transitivity of G on 12 points), and so $S \neq T_0$. This forces $|T : S| \leq 2$, contrary to 2.5.

Finally, assume $G \cong Ru$. By the *Atlas* G has two classes of involutions, one of which is 2-central and the other having centralizer isomorphic to $E_4 \times Sz(8)$. In particular, involutions centralizing an element of order 3 or whose centralizers have 2-rank > 5 are all 2-central. As usual, S contains a 2-central involution, hence contains all involutions in T that are 2-central in G.

Let H be a subgroup of G isomorphic to $E_{2^6}G_2(2)$ (split extension) with $H \cap T \in Syl_2(H)$. Note that all involutions in $O_2(H)$ centralize the elementary abelian subgroup $O_2(H)$ of rank 6, so all are 2-central in G; in particular, $O_2(H) \leq S$. Let K be a subgroup of H isomorphic to $G_2(2) = U_3(3) : 2$. By properties of $G_2(2)$, the involutions in $K - K'$ centralize elements of order 3 in K', hence all are 2-central in G. Thus $H \cap S$ contains elements of $K - K'$. It follows that $H = \langle S^H \rangle$. By induction, S contains a Sylow 2-subgroup of H, i.e., H is an S-group. By 2.5 and order considerations we then obtain:

$$S \in Syl_2(H) \quad \text{and} \quad |T : S| = 4. \tag{2.10.1}$$

Now let P be the maximal 2-local subgroup of G containing T with $P = QL$, $Q = O_2(P)$ of order 2^{11} and $L \cong L_3(2)$. Let $P_0 = \langle S^P \rangle$ so that by induction $S \in Syl_2(P_0)$. Since by (2.10.1) $S \not\leq Q$, it follows that

$$L \leq P_0 \quad \text{and} \quad |P : P_0| = 4. \tag{2.10.2}$$

Thus $S \cap Q$ is a normal subgroup of P of index 4 in Q. Since L acts on $Q/S \cap Q$, we must have $[Q, L] \leq S \cap Q$.

By the *Atlas* there is one G-conjugacy class of elements of order 7, and the centralizer of an element of order 7 has order 28. In the centralizer of a non-2-central involution (described above) one sees a subgroup $E_4 \times Z_7$ where all involutions are non-2-central (the E_4 is the centralizer of the Suzuki component). Thus all involutions commuting with an element of order 7 are of non-2-central type.

Let R be a subgroup of L of order 21. Now $Q = C_Q(R)[Q, R]$, and so by the preceding paragraph $|Q : [Q, R]| \leq 4$. Since, however, $[Q, R] \leq [Q, L] \leq P_0$, by (2.10.2) we must have

$$|Q : [Q, R]| = 4 \quad \text{and} \quad |C_Q(R)| = 4.$$

This leads to a contradiction because, as noted above, $C_Q(R)$ — which is contained in the centralizer of the subgroup R' of order 7 — is generated by non-2-central involutions. But non-2-central involutions do not commute with any element of order 3, such as those in R. This contradiction shows that G is not isomorphic to Ru, and so completes the elimination of all sporadic groups as candidates for the counterexample G.

It remains to eliminate the groups of Lie type over fields of odd order. In order to do so we use the fact that the B-conjecture holds in G, so that 2-components in centralizers of involutions are quasisimple. Let $\mathcal{L}(G)$ be the set of 2-components of centralizers of involutions of G.

2.11. *If $L \in \mathcal{L}(G)$, then some G-conjugate of S contains a Sylow 2-subgroup of L; in other words, every element of $\mathcal{L}(G)$ is an S-group.*

Suppose this is not the case. Let t be an involution in G and let L be a component of $C_G(t)$ of maximal order which is not an S-group; subject to this choose t and L so that $C_G(t)$ contains a subgroup S_0 of largest order which is conjugate to a subgroup of S. Replacing L and t by conjugates if necessary, we may assume $S_0 = C_S(t)$. Note that $S_0 \neq 1$. Let $L^* = \langle L^{S_0} \rangle$, so L^* is a product of commuting S_0-conjugates of L. Since we have reduced to when G is a group of Lie type over a field of odd order, by 14–1 of [12] L is also a group of Lie type over a field of odd order. Thus by induction in S_0L^* we obtain that S_0 is a Sylow 2-subgroup of $\langle S_0^{L^*} \rangle$. Since L is not an S-group we must have that $S_0 = \langle S_0^{L^*} \rangle \trianglelefteq S_0L^*$. Now $[S_0, L^*] \leq S_0 \cap L^* \leq Z(L^*)$. Since L^* is perfect, it follows from the Three Subgroups Lemma that

$$[S_0, L^*] = 1. \tag{2.11.1}$$

Theorem B1 of [10] asserts that for any strongly closed 2-subgroup S of G we have $C_G(S)^{(\infty)} \trianglelefteq G$. By (2.11.1) therefore $S_0 \neq S$. But now for some involution u in S_0, $C_S(u) > S_0$. Moreover, since u centralizes L, the L-balance Theorem (e.g., Lemma 2.7 of [1]) asserts that L pumps up in $C_G(u)$ to a component or product of two components where each of these pumpup components has order at least $|L|$. Furthermore, since L is not an S-group, neither are these pumpups. The maximality of t and L are therefore violated. This contradiction completes the proof of 2.11.

2.12. *G is not isomorphic to $L_4(q)$, $U_4(q)$ or $S_4(q)$, for any odd q.*

By the *Atlas*, $U_4(3)$ contains a 2-constrained subgroup isomorphic to $E_{16}A_6$ which is of odd index, hence G is not isomorphic to $U_4(3)$ by 2.7. The *Atlas* also indicates that $L_4(3)$ contains a subgroup isomorphic to $U_4(2):2$ of odd index, so it follows by induction and 2.9 that G is not of this type either. The group $S_4(3) \cong U_4(2)$ was eliminated as a contender for G in 2.9.

Consider next when G is isomorphic to $L_4(q)$, $U_4(q)$ or $S_4(q)$ for some $q > 3$. Let z be an involution in $Z(T)$. In all these groups $C_G(z)$ is of the form $LL^xD\langle x \rangle$, where L and L^x are components of type $SL_2(q)$ interchanged by x, D is a cyclic group (of diagonal matrices), and x may be chosen to be conjugate to z. By 2.11, S contains a Sylow 2-subgroup of $LL^x\langle x \rangle$. Thus T/S is cyclic, and so 2.5 gives the desired contradiction.

2.13. *G is not isomorphic to $O_7(q)$, $O_8^+(q)$ or $O_8^-(q)$, for any odd q.*

First note that by the *Atlas*, we have the following containments:

$$O_7(3) \geq S_6(2), \qquad O_8^+(3) \geq O_8^+(2), \qquad O_8^-(3) \geq O_7(3) \cdot 2.$$

In all cases the index of the subgroup is odd. It follows by induction that each subgroup is an S-group, and hence $S = T$, a contradiction. Thus we may assume $q > 3$.

In each of the remaining orthogonal groups under consideration let z be an involution in G that lifts to an element of order 2 in the spinor covering group of G; there is a unique such class in G, and z is 2-central in G (see, for example, Lemmas 4.10 – 4.14 in [14] for the cited properties of these orthogonal groups). The centralizer in G of z contains a subgroup H of index 2 such that $H = E(C_G(z))W$, where $\langle z \rangle = Z(E(C_G(z)))$ and W is generated by some G-conjugates of z. (More precisely, $E(C_{O_7(q)}(z)) \cong SL_2(q)^2 \times L_2(q)$, $E(C_{O_8^+(q)}(z)) \cong SL_2(q)^4$, and $E(C_{O_8^-(q)}(z)) \cong SL_2(q)^2 \times L_2(q^2)$, where the repeated $SL_2(q)$ factors are central products with a common center.) By 2.11, $E(C_G(z))$ is an S-group. It then follows from the generation property of W that G contains an S-subgroup whose index is twice an odd number. This violates 2.5 and so completes the proof of 2.13.

2.14. *G is not isomorphic to a Chevalley group over a field of odd order.*

Suppose to the contrary G is isomorphic to a Chevalley group (untwisted or twisted) over $\mathbb{F}_q$, where q is a power of an odd prime p. The families $L_2(q)$, $G_2(q)$, ${}^3D_4(q)$, $L_3(q)$, and $U_3(q)$ all have one class of involutions and are easily eliminated by the second sentence of 2.6 — see [11] for the needed properties. The families $Re(q)$ and $O_k^{\pm}(q)$, $k = 5$, 6, 7, 8 have already been eliminated by 2.4, 2.12 and 2.13. Thus we may assume G does not belong to any of these families. In the notation of [2] and [5], let J be a *fundamental subgroup* of G, so $J \cong SL_2(q)$ and J is a component or solvable component of $C_G(z)$, where $\langle z \rangle = Z(J)$. We may choose J so that $N_T(J) \in Syl_2(N_G(J))$. Let $R = J \cap T \in Syl_2(J)$. Following [3], $Fun(T)$ is the set of G-conjugates of R that lie in T. It is shown in [2] that distinct elements of $Fun(T)$ commute; moreover, if $Fun(T) = \{R, R_2, \ldots, R_m\}$ and R_i is a Sylow 2-subgroup of the conjugate J_i of J, then $JJ_2 \ldots J_m$ is a central product of these conjugates of J. Let $M = N_G(\langle Fun(T) \rangle)$ i.e., M is the normalizer of the weak closure of R in T with respect to G. Note that $T \leq M$. For each of the Chevalley groups the (transitive) permutation action, Δ, of M on $Fun(T)$ is described in [3] — see Table III below. Finally, it is shown in [5] that

$$N_G(J) = JXH$$

where $[J, X] = 1$, X is a Levi component of the maximal parabolic which is the normalizer of a Sylow p-subgroup of J, and H is a Cartan subgroup of that parabolic. For each Chevalley group the isomorphism type of X is also described in [5] — see Table III below. In all cases under consideration X is generated by G-conjugates of J. Our aim is to first show that S contains a Sylow 2-subgroup of $N_G(J)$, or equivalently that $N_G(J)$ is an S-group. To do this it suffices to demonstrate that each of J, X and H are S-groups.

Since no composition factor of X is of type $U_3(2^n)$ or $Sz(2^n)$ and since $S \cap X \not\leq Z(X)$, it follows that S contains a Sylow 2-subgroup of X, i.e., X is an S-group. Since X is generated by G-conjugates of J we also get that J is an S-group (which we knew already from 2.11 when $q > 3$).

Table III

G	X	$\lvert Fun(T)\rvert$	Δ
$L_{2n+a}(q)$, $a=0,1$	$SL_{2n+a-2}(q)$	n	Σ_n
$S_{2n}(q)$	$Sp_{2n-2}(q)$	n	Σ_n
$U_{2n+a}(q)$, $a=0,1$	$SU_{2n+a-2}(q)$	n	Σ_n
$O^{\pm}_{4n+a}(q)$, $a=1,2,3$	$SL_2(q)\Omega^{\pm}_{4n+a-4}(q)$	$2n$	$E_{2^n}\Sigma_n$
$O^{-}_{4n+4}(q)$	$SL_2(q)\Omega^{-}_{4n}(q)$	$2n$	$E_{2^n}\Sigma_n$
$O^{+}_{4n}(q)$	$SL_2(q)\Omega^{+}_{4n-4}(q)$	$2n$	$E_{2^{n-1}}\Sigma_n$
$F_4(q)$	$Sp_6(q)$	4	S_4
$E_6(q)$	$SL_6(q)/Z_{(q-1,3)}$	4	S_4
${}^2E_6(q)$	$SU_6(q)/Z_{(q+1,3)}$	4	S_4
$E_7(q)$	$\widehat{O}^{+}_{12}(q)$	7	$L_3(2)$
$E_8(q)$	$E_7(q)$	8	$E_8L_3(2)$

Now H is a Cartan subgroup of G and so H is generated by cyclic subgroups H_α (isomorphic to copies of the multiplicative groups of fields), one for each root length. Each of these rank 1 tori are G-conjugate to a subgroup of the (S-group) Levi component X, hence each H_α is an S-group. (In the case where $G \cong O^{\pm}_k(q)$, the component X pumps up to a maximal component of type $O^{\pm}_{k-2}(q)$ in the centralizer of a rank 2 involution; the pumpup of X contains a conjugate of these tori.) Since H is generated by commuting S-groups, it too is an S-group. Putting these generational facts together proves:

$$S \text{ contains a Sylow 2-subgroup of } N_G(J). \tag{2.14.1}$$

We derive the final contradiction by showing M contains a subgroup of index at most 2 which is an S-group. Let N be the kernel of the permutation representation of M on $Fun(T)$, and let $\overline{M} = M/N = \Delta$. Note that $\overline{N_M(J)}$ is the stabilizer of a point in this action. Let $S_0 = S \cap N_M(J)$ so that

$$\overline{S_0} \text{ is a Sylow 2-subgroup of the stabilizer in } \Delta \text{ of a point.}$$

Let n be the parameter in Table III above which relates to the degree of the permutation representation of $\overline{M} = \Delta$ on $Fun(T)$. Consider first when $n \geq 3$. In these cases the stabilizer of a point has even order, i.e., $\overline{S_0} \neq \overline{1}$, and $\overline{M} = \langle \overline{S_0}^{\overline{M}} \rangle$. Since no composition factor of $\overline{M}$ is of type $U_3(2^n)$ or $Sz(2^n)$, by induction

$$\overline{S} \text{ is a Sylow 2-subgroup of } \overline{M}. \tag{2.14.2}$$

Since $N \leq N_M(J)$, by (2.14.1) and (2.14.2) we obtain $S = T$, a contradiction.

Finally, consider when $n = 2$. By 2.13, $\Delta \neq E_2\Sigma_2$. Thus

$$|\overline{M} : \langle \overline{S_0}^{\overline{M}} \rangle| = 2.$$

As in the preceding paragraph, it follows that that $|T : S| \leq 2$, and now 2.5 is violated. This contradiction establishes 2.14.

This completes the elimination of all finite simple groups as possible isomorphism types for G, and so establishes Theorem 1.

3. Proof of the Theorem 2

Assume the hypothesis of Theorem 2 and proceed by induction on the order of G. In the notation of the theorem S is a strongly closed 2-subgroup which is not a Sylow 2-subgroup of its normal closure in G. It follows from Lemma 1(2) and the remarks preceding the statement of Theorem 2 that $G/\mathcal{O}_S(G)$ also satisfies the hypotheses of Theorem 2. If $\mathcal{O}_S(G) \neq 1$, then by induction $\langle S^G \rangle \mathcal{O}_S(G)/\mathcal{O}_S(G)$ is a direct product of Suzuki and unitary groups with S projecting onto the center of one of its Sylow 2-subgroups. Theorem 2 is then valid in this case, so we may assume

$$\mathcal{O}_S(G) = 1. \tag{3.1}$$

Let S_0 be a minimal strongly closed subgroup of S. By Theorem 1, $\langle S_0^G \rangle = L$ is a direct product of simple groups isomorphic to $U_3(2^n)$ or $Sz(2^n)$ permuted transitively by G, and $S \cap L$ is the center of some Sylow 2-subgroup of L. Let T be a Sylow 2-subgroup of G containing S, let $T_0 = T \cap L$, let $Z = Z(T_0)$, and let $B = N_L(T_0)$. Thus B is a direct product of Borel subgroups and $B = T_0 H$ where H is the product of the (odd order) Cartan subgroups in B. We argue that

$$S = (S \cap L) \times (S \cap C_G(L)). \tag{3.2}$$

To see this, first note that S normalizes B and $[S, T_0] \leq S \cap B = Z$. Thus $S \leq C_{SB}(T_0/Z) \trianglelefteq SB$. Since S is strongly closed and contained in $O_2(SB)$, H acts on S. Now $[S, H] \leq S \cap B = Z$. Moreover, by properties of the Suzuki and unitary groups, H has no nontrivial fixed points on Z. The Fitting decomposition of H acting on S thus gives that $S = C_S(H)Z$ and $C_S(H) \cap Z = 1$. Let $S_1 = C_S(H)$. Since S_1 centralizes H, it normalizes each component of L. Since the involutory outer automorphisms of $Sz(2^n)$ and $U_3(2^n)$ are all of field type, a nontrivial 2-power automorphism of any component of L does not centralize a full Cartan subgroup of that component. Thus S_1 must act trivially on L, whence $S_1 = C_S(L)$ and (3.2) holds.

If $S_1 = 1$, the conclusion to Theorem 2 holds; so assume $S_1 \neq 1$. Now $S_1 = S \cap C_G(L)$ is strongly closed in G. By Frattini's Argument $G = C_G(L)N_G(S_1)$, so the normal closure of S_1 in G is the same as its normal closure in $C_G(L)$. It follows from (3.1) that $\mathcal{O}_{S_1}(C_G(L)) =$

1, whence S_1 is not a Sylow 2-subgroup of its normal closure. Applying induction to S_1 in the proper subgroup $C_G(L)$ and using (3.2) now completes the proof of Theorem 2.

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