ADAPTIVE WAVELET PACKETS - BEST BASIS SELECTION

The wavelet packet transform is a generalization of the orthogonal discrete wavelet transform. Wavelet packet vectors for generated for a “full binary tree”. For a length N signal, there are $2^{N/2}$ possible orthogonal bases. Best basis can be generated using a number of cost functions.

These notes are taken from MATLAB Help. You can find these notes by searching for “Choosing for the optimal decomposition”.

Choosing the Optimal Decomposition

Based on the organization of the wavelet packet library, it is natural to count the decompositions issued from a given orthogonal wavelet.

A signal of length $N = 2L$ can be expanded in different ways, where $L$ is the number of binary subtrees of a complete binary tree of depth $L$. As a result, (see [Mal98] page 323).

As this number may be very large, and since explicit enumeration is generally unmanageable, it is interesting to find an optimal decomposition with respect to a convenient criterion, computable by an efficient algorithm. We are looking for a minimum of the criterion.

Functions verifying an additivity-type property are well suited for efficient searching of binary-tree structures and the fundamental splitting. Classical entropy-based criteria match these conditions and describe information-related properties for an accurate representation of a given signal. Entropy is a common concept in many fields, mainly in signal processing. Let us list four different entropy criteria (see [CoiW92]); many others are available and can be easily integrated (type help wentropy). In the following expressions $s$ is the signal and $(s_i)$ are the coefficients of $s$ in an orthonormal basis.

The entropy $E$ must be an additive cost function such that $E(0) = 0$ and The (nonnormalized) Shannon entropy

$$E(s) = \sum_i E(s_i)$$

(1.) The (nonnormalized) Shannon entropy

$$E_1(s_i) = -s_i^2 \log(s_i^2)$$

so

$$E_1(s) = -\sum_i s_i^2 \log(s_i^2)$$

with the convention that $0 \log(0) = 0$.

(2.) The concentration in $l_p$ norm with $1 \leq p$

$$E_2(s_i) = |s_i|^p$$
so

\[ E_2(s) = \sum_i |s_i|^p = ||s||^p \]

(3.) The logarithm of the "energy" entropy

\[ E_3(s_i) = \log(s_i^2) \]

so

\[ E_3(s) = \sum_i \log(s_i^2) \]

with the convention \( \log(0) = 0 \).

(4.) The threshold entropy

\[ E_4(s_i) = 1 \text{ if } |s_i| > \epsilon \]

and 0 elsewhere, so

\[ E_4(s) = \text{ total number } \{i, \text{ where } |s_i| > \epsilon\} \]

is the number of time instants when the signal is greater than a threshold \( \epsilon \).

These entropy functions are available using the wentropy M-file.

Example 1. Compute Various Entropies.
Generate a signal of energy equal to 1.
\( s = \text{ones}(1,16)*0.25; \)
Compute the Shannon entropy of \( s \).
\( e1 = \text{wentropy}(s,\text{’shannon’}) \)
\( e1 = 2.7726 \)
Compute the l1.5 entropy of \( s \), equivalent to \( \text{norm}(s,1.5)^{1.5} \).
\( e2 = \text{wentropy}(s,\text{’norm’,1.5}) \)
\( e2 = 2 \)
Compute the "log energy" entropy of \( s \).
\( e3 = \text{wentropy}(s,\text{’log energy’}) \)
\( e3 = -44.3614 \)
Compute the threshold entropy of \( s \), using a threshold value of 0.24.
\( e4 = \text{wentropy}(s,\text{’threshold’, 0.24}) \)
\( e4 = 16 \)
Example 2: Minimum-Entropy Decomposition.

This simple example illustrates the use of entropy to determine whether a new splitting is of interest to obtain a minimum-entropy decomposition.

1.) We start with a constant original signal. Two pieces of information are sufficient to define and to recover the signal (i.e., length and constant value).
\[ w_{00} = \text{ones}(1,16) \ast 0.25; \]

2.) Compute entropy of original signal.
\[ e_{00} = \text{wentropy}(w_{00},\text{'shannon'}) \]
\[ e_{00} = 2.7726 \]

3.) Then split \( w_{00} \) using the haar wavelet.

\[ [w_{10},w_{11}] = \text{dwt}(w_{00},\text{'db1'}); \]

4.) Compute entropy of approximation at level 1.
\[ e_{10} = \text{wentropy}(w_{10},\text{'shannon'}) \]
\[ e_{10} = 2.0794 \]

The detail of level 1, \( w_{11} \), is zero; the entropy \( e_{11} \) is zero. Due to the additivity property the entropy of decomposition is given by \( e_{10} + e_{11} = 2.0794 \). This has to be compared to the initial entropy \( e_{00} = 2.7726 \). We have \( e_{10} + e_{11} < e_{00} \), so the splitting is interesting.

5.) Now split \( w_{10} \) (not \( w_{11} \) because the splitting of a null vector is without interest since the entropy is zero).

\[ [w_{20},w_{21}] = \text{dwt}(w_{10},\text{'db1'}); \]

6.) We have \( w_{20} = 0.5 \ast \text{ones}(1,4) \) and \( w_{21} \) is zero. The entropy of the approximation level 2 is
\[ e_{20} = \text{wentropy}(w_{20},\text{'shannon'}) \]
\[ e_{20} = 1.3863 \]

7.) Again we have \( e_{20} + 0 < e_{10} \), so splitting makes the entropy decrease. Then

\[ [w_{30},w_{31}] = \text{dwt}(w_{20},\text{'db1'}); \]
\[ e_{30} = \text{wentropy}(w_{30},\text{'shannon'}) \]
\[ e_{30} = 0.6931 \]

\[ [w_{40},w_{41}] = \text{dwt}(w_{30},\text{'db1'}); \]
\[ w_{40} = 1.0000 \]
\[ w_{41} = 0 \]

\[ e_{40} = \text{wentropy}(w_{40},\text{'shannon'}) \]
\[ e_{40} = 0 \]

In the last splitting operation we find that only one piece of information is needed to reconstruct the original signal. The wavelet basis at level 4 is a best basis according to Shannon entropy (with null optimal entropy since \( e_{40} + e_{41} + e_{31} + e_{21} + e_{11} = 0 \)).
(8.) Perform wavelet packets decomposition of the signal s defined in Example 1.

\[ t = \text{wpdec}(s,4,\text{’haar’,’shannon’}); \]

The wavelet packet tree is shown below. It is also shown with the nodes labeled with o in Figure 6-42, MATLAB Help, with nodes labeled with the original entropy numbers.

(9.) Now compute the best tree.

\[ \text{bt} = \text{besttree}(t); \]

The best tree is displayed in figure 6-43 in MATLAB Help and below. The best tree corresponds to the dyadic wavelet tree. Label the nodes with the optimal entropy.