1. Representation of convolution using matrices

We have, for a LTI discrete-time system

\[ y[n] = \sum h[m]x[n-m]. \]

We can represent convolution using matrices. Assume that the desired matrix structure is as shown below. That is, for convenience, we assume \( x \) to be a \( 4 \times 1 \) vector although it is typically much longer. Fill in the blanks for each of the filter matrices. You can assume the filter size to be 3.

(i) Assume first that the filter is causal, consisting of \( \{h(0), h(1), h(2)\} \).

\[
\begin{bmatrix}
  y_0 \\
  y_1 \\
  y_2 \\
  y_3
\end{bmatrix} =
\begin{bmatrix}
  - & - & - & - \\
  - & - & - & - \\
  - & - & - & - \\
  - & - & - & -
\end{bmatrix}
\begin{bmatrix}
  x_0 \\
  x_1 \\
  x_2 \\
  x_3
\end{bmatrix}
\]

(ii) Here the filter is non-causal, that is, it has elements \( \{h(-2), h(-1), h(0)\} \).

\[
\begin{bmatrix}
  y_0 \\
  y_1 \\
  y_2 \\
  y_3
\end{bmatrix} =
\begin{bmatrix}
  - & - & - & - \\
  - & - & - & - \\
  - & - & - & - \\
  - & - & - & -
\end{bmatrix}
\begin{bmatrix}
  x_0 \\
  x_1 \\
  x_2 \\
  x_3
\end{bmatrix}
\]

(iii) Here the filter is causal again.

\[
\begin{bmatrix}
  y_0 \\
  y_1 \\
  y_2 \\
  y_3
\end{bmatrix} =
\begin{bmatrix}
  - & - & - & - \\
  - & - & - & - \\
  - & - & - & - \\
  - & - & - & -
\end{bmatrix}
\begin{bmatrix}
  x_{-2} \\
  x_{-1} \\
  x_0 \\
  x_1
\end{bmatrix}
\]
2. (10) Inverse without filter banks

Consider the cascade of 2 filters $h_0[n]$ and $h_1[n]$ as shown above:

(a) Give the time domain condition on $h_0[n]$, $h_1[n]$ such that $h_1[n]$ is the inverse of $h_0[n]$.

(b) Same as (a) above, except give it in the z-transform domain.

(c) Let $H_0(z) = 1 + 2z^{-1}$. Find the inverse filter $H_1(z)$. $H_0(z)$ is FIR. What type of a filter is $H_1(z)$?

(d) In (c) above, let $x_1[n], y_1[n]$ be the input and output signals to $h_1[n]$. Give the difference equation relating $x_1[n]$ and $y_1[n]$. Is this a recursive or non-recursive difference equation. Hence, what type of difference equation is represented by $H_1(z)$?
3. **Inverse with filter banks**

Consider the 2-channel filterbank shown, where $\mathbf{x}$, $\mathbf{y}$, $\mathbf{v}_0$, $\mathbf{v}_1$ are vectors.

(a) Assume that we have the analysis equation $\mathbf{c} = \mathbf{A} \mathbf{x}$ where

\[
\mathbf{c} = \begin{bmatrix} v_0 \\ v_1 \end{bmatrix}
\]

Express $\mathbf{A}$ in terms of the filter sub-matrices $\mathbf{C}$ and $\mathbf{D}$ written as $\downarrow 2 \mathbf{C}$ and $\downarrow 2 \mathbf{D}$. Refer to the latter two symbols as $\mathbf{L}$ and $\mathbf{B}$ respectively.

(b) Assume that the analysis matrix $\mathbf{A}$ is orthogonal. \{ $A^{-1} = A^T$ \}. Write $\mathbf{x}$ in terms of $\mathbf{c}$ using $\mathbf{L}$ and $\mathbf{B}$.

(c) Explain, justify, that if $\mathbf{L}$ and $\mathbf{B}$ correspond to FIR filters, then for the orthogonal case, the synthesis filters are also FIR.
4. Perfect reconstruction with Haar filters

(a) Now we see how orthogonality and PR manifest for a simple filter bank. Generate the 2-channel Haar filters:

\[ [H_0, H_1, F_0, F_1] = wfilters('haar') \]

Note that these are all meant to be causal filters. Confirm the orthogonality of the analysis matrix.

(b) Show that

\[ H_0(z)F_0(z) + H_1(z)F_1(z) = 2z^{-m} \]

for some integer \( m \). Show that the aliasing condition is satisfied. What is \( m \) and hence what is \( y[n] \) in terms of \( x[n] \)?
5. Fast wavelet transform

(a) Assume that $H_0$ and $H_1$ are both FIR filters of size, say, 3. Consider the 2-level decomposition shown. Write the output wavelet coefficients $c$ in terms of the input vector $x$, using $L$, $B$ notation for a 1-level decomposition. Then do the same, as a product of 2 matrices, for the 2-level decomposition.

(b) For an input vector $x$ of size $2^N$ which allows a $N$-level decomposition, show that the computation (multiplication) cost for determining $c$, that is doing a $N$ - level 1-D dwt, is of order $N$, $O(N)$. Note that,

$$\sum_{k=0}^{N} \left(\frac{1}{2}\right)^k < 2$$