Types of Transfer Functions

- The time-domain classification of an LTI digital transfer function sequence is based on the length of its impulse response:
  - Finite impulse response (FIR) transfer function
  - Infinite impulse response (IIR) transfer function

In the case of digital transfer functions with frequency-selective frequency responses, there are two types of classifications:

1. Classification based on the shape of the magnitude function $|H(e^{j\omega})|
2. Classification based on the form of the phase function $\theta(\omega)$

Classification Based on Magnitude Characteristics

- One common classification is based on an ideal magnitude response
- A digital filter designed to pass signal components of certain frequencies without distortion should have a frequency response equal to one at these frequencies, and should have a frequency response equal to zero at all other frequencies

Ideal Filters

- The range of frequencies where the frequency response takes the value of one is called the passband
- The range of frequencies where the frequency response takes the value of zero is called the stopband

Ideal Filters

- Frequency responses of the four popular types of ideal digital filters with real impulse response coefficients are shown below:
- Lowpass filter: Passband - $0 \leq \omega \leq \omega_c$
  Stopband - $\omega < \omega_c$ $\leq \pi$
- Highpass filter: Passband - $\omega_c \leq \omega \leq \pi$
  Stopband - $0 \leq \omega < \omega_c$
- Bandpass filter: Passband - $\omega_{c1} \leq \omega \leq \omega_{c2}$
  Stopband - $0 \leq \omega < \omega_{c1}$ and $\omega_{c2} < \omega \leq \pi$
- Bandstop filter: Stopband - $\omega_{c1} \leq \omega \leq \omega_{c2}$
  Passband - $0 \leq \omega \leq \omega_{c1}$ and $\omega_{c2} \leq \omega \leq \pi$
Ideal Filters

• The frequencies $\omega_1$, $\omega_2$, and $\omega_3$ are called the cutoff frequencies.

• An ideal filter has a magnitude response equal to one in the passband and zero in the stopband, and has a zero phase everywhere.

Earlier in the course we derived the inverse DTFT of the frequency response of the ideal lowpass filter:

$$h_{LP}[n] = \frac{\sin \omega_1 n}{\pi n}, \quad -\infty < n < \infty$$

We have also shown that the above impulse response is not absolutely summable, and hence, the corresponding transfer function is not BIBO stable.

Ideal Filters

• Also, $h_{LP}[n]$ is not causal and is of doubly infinite length.

• The remaining three ideal filters are also characterized by doubly infinite, noncausal impulse responses and are not absolutely summable.

• Thus, the ideal filters with the ideal “brick wall” frequency responses cannot be realized with finite dimensional LTI filter.

Moreover, the magnitude response is allowed to vary by a small amount both in the passband and the stopband.

Typical magnitude response specifications of a lowpass filter are shown below.

Bounded Real Transfer Functions

• A causal stable real-coefficient transfer function $H(z)$ is defined as a bounded real (BR) transfer function if

$$|H(e^{j\omega})| \leq 1 \quad \text{for all values of } \omega$$

• Let $x[n]$ and $y[n]$ denote, respectively, the input and output of a digital filter characterized by a BR transfer function $H(z)$ with $X(e^{j\omega})$ and $Y(e^{j\omega})$ denoting their DTFTs.
Bounded Real Transfer Functions

• Then the condition $|H(e^{j\omega})| \leq 1$ implies that $|Y(e^{j\omega})|^2 \leq |X(e^{j\omega})|^2$

• Integrating the above from $-\pi$ to $\pi$, and applying Parseval’s relation we get

$$\sum_{n=-\infty}^{\infty} |y[n]|^2 \leq \sum_{n=-\infty}^{\infty} |x[n]|^2$$

Bounded Real Transfer Functions

• Thus, for all finite-energy inputs, the output energy is less than or equal to the input energy implying that a digital filter characterized by a BR transfer function can be viewed as a passive structure

• If $|H(e^{j\omega})| = 1$, then the output energy is equal to the input energy, and such a digital filter is therefore a lossless system

Bounded Real Transfer Functions

• A causal stable real-coefficient transfer function $H(z)$ with $|H(e^{j\omega})| = 1$ is thus called a lossless bounded real (LBR) transfer function

• The BR and LBR transfer functions are the keys to the realization of digital filters with low coefficient sensitivity

• Example – Consider the causal stable IIR transfer function

$$H(z) = \frac{K}{1 - \alpha z^{-1}}, \quad 0 < |\alpha| < 1$$

where $K$ is a real constant

• Its square-magnitude function is given by

$$|H(e^{j\omega})|^2 = |H(z)H(z^{-1})|_{z=e^{j\omega}} = \frac{K^2}{(1 + \alpha^2) - 2\alpha \cos \omega}$$

Bounded Real Transfer Functions

• The maximum value of $|H(e^{j\omega})|^2$ is obtained when $2\alpha \cos \omega$ in the denominator is a maximum and the minimum value is obtained when $2\alpha \cos \omega$ is a minimum

• For $\alpha > 0$, maximum value of $2\alpha \cos \omega$ is equal to $2\alpha$ at $\omega = 0$, and minimum value is $-2\alpha$ at $\omega = \pi$

• Thus, the maximum value of $|H(e^{j\omega})|^2$ is equal to $K^2/(1 - \alpha^2)$ at $\omega = 0$

• The maximum value can be made equal to 1 by choosing $K = 1 - \alpha$

• Hence, $H(z) = \frac{1 - \alpha}{1 - \alpha z^{-1}}, \quad 0 < \alpha < 1$

is a BR function for $\alpha > 1$
Allpass Transfer Function

Definition
• An IIR transfer function $A(z)$ with unity magnitude response for all frequencies, i.e.,
  \[ |A(e^{j\omega})|^2 = 1, \quad \text{for all } \omega \]
is called an allpass transfer function
• An $M$-th order causal real-coefficient allpass transfer function is of the form
  \[ A_M(z) = \frac{d_M + d_{M-1}z^{-1} + \cdots + d_{1}z^{-M+1} + z^{-M}}{1 + d_1z^{-1} + \cdots + d_{M-1}z^{-M+1} + d_Mz^{-M}} \]

Allpass Transfer Function

• If we denote the denominator polynomials of $A_M(z)$ as $D_M(z)$:
  \[ D_M(z) = 1 + d_1z^{-1} + \cdots + d_{M-1}z^{-M+1} + d_Mz^{-M} \]
then it follows that $A_M(z)$ can be written as:
  \[ A_M(z) = \pm \frac{z^{-M}D_M(z^{-1})}{D_M(z)} \]
• Note from the above that if $z = re^{j\phi}$ is a pole of a real coefficient allpass transfer function, then it has a zero at $z = \frac{1}{r}e^{-j\phi}$

Allpass Transfer Function

• The numerator of a real-coefficient allpass transfer function is said to be the mirror-image polynomial of the denominator, and vice versa
• We shall use the notation $\tilde{D_M}(z)$ to denote the mirror-image polynomial of a degree-$M$ polynomial $D_M(z)$, i.e.,
  \[ \tilde{D_M}(z) = z^{-M}D_M(z^{-1}) \]

Allpass Transfer Function

• To show that $|A_M(e^{j\omega})| = 1$ we observe that
  \[ A_M(z^{-1}) = \pm \frac{z^{-M}D_M(z)}{D_M(z^{-1})} \]
• Therefore
  \[ A_M(z)A_M(z^{-1}) = \frac{z^{-M}D_M(z^{-1})z^MD_M(z)}{D_M(z)D_M(z^{-1})} \]
• Hence
  \[ |A_M(e^{j\omega})|^2 = A_M(z)A_M(z^{-1}) \bigg|_{z = e^{j\omega}} = 1 \]

Allpass Transfer Function

• Now, the poles of a causal stable transfer function must lie inside the unit circle in the $z$-plane
• Hence, all zeros of a causal stable allpass transfer function must lie outside the unit circle in a mirror-image symmetry with its poles situated inside the unit circle
Allpass Transfer Function

- Figure below shows the principal value of the phase of the 3rd-order allpass function
  \[ A_3(z) = -0.2 + 0.18z^{-1} + 0.4z^{-2} + z^{-3} \]
  \[ 1 + 0.4z^{-1} + 0.18z^{-2} - 0.2z^{-3} \]
- Note the discontinuity by the amount of 2π in the phase \( \theta(\omega) \)

Allpass Transfer Function

- If we unwrap the phase by removing the discontinuity, we arrive at the unwrapped phase function \( \theta_c(\omega) \) indicated below
- Note: The unwrapped phase function is a continuous function of \( \omega \)

Allpass Transfer Function

- The unwrapped phase function of any arbitrary causal stable allpass function is a continuous function of \( \omega \)

Properties

- (1) A causal stable real-coefficient allpass transfer function is a lossless bounded real (LBR) function or, equivalently, a causal stable allpass filter is a lossless structure

- (2) The magnitude function of a stable allpass function \( A(z) \) satisfies:
  \[ \begin{cases} 
  < 1, & \text{for } |z| > 1 \\
  = 1, & \text{for } |z| = 1 \\
  > 1, & \text{for } |z| < 1 
  \end{cases} \]

- (3) Let \( \tau(\omega) \) denote the group delay function of an allpass filter \( A(z) \), i.e.,
  \[ \tau(\omega) = -\frac{d}{d\omega} \theta_c(\omega) \]

Allpass Transfer Function

- The unwrapped phase function \( \theta_c(\omega) \) of a stable allpass function is a monotonically decreasing function of \( \omega \) so that \( \tau(\omega) \) is everywhere positive in the range \( 0 < \omega < \pi \)

- The group delay of an \( M \)-th order stable real-coefficient allpass transfer function satisfies:
  \[ \pi \int_{0}^{\pi} \tau(\omega)d\omega = M\pi \]

A Simple Application

- A simple but often used application of an allpass filter is as a delay equalizer
- Let \( G(z) \) be the transfer function of a digital filter designed to meet a prescribed magnitude response
- The nonlinear phase response of \( G(z) \) can be corrected by cascading it with an allpass filter \( A(z) \) so that the overall cascade has a constant group delay in the band of interest
### Allpass Transfer Function

- Since \( |A(e^{j\omega})| = 1 \), we have
  \[ |G(e^{j\omega})A(e^{j\omega})| = |G(e^{j\omega})| \]
- Overall group delay is given by the sum of the group delays of \( G(z) \) and \( A(z) \)

**Example** – Figure below shows the group delay of a 4th order elliptic filter with the following specifications:
- \( \omega_p = 0.3\pi \)
- \( \delta_p = 1 \) dB, \( \delta_s = 35 \) dB

![Group Delay of 4th Order Elliptic Filter](image)

### Classification Based on Phase Characteristics

- A second classification of a transfer function is with respect to its phase characteristics
- In many applications, it is necessary that the digital filter designed does not distort the phase of the input signal components with frequencies in the passband

**Figure** below shows the group delay of the original elliptic filter cascaded with an 8th order allpass section designed to equalize the group delay in the passband

![Group Delay Equalization](image)

### Zero-Phase Transfer Function

- One way to avoid any phase distortion is to make the frequency response of the filter real and nonnegative, i.e., to design the filter with a **zero phase characteristic**
- However, it is not possible to design a causal digital filter with a zero phase

**For non-real-time processing of real-valued input signals of finite length, zero-phase filtering can be very simply implemented by relaxing the causality requirement**

- One zero-phase filtering scheme is sketched below

\[
\begin{align*}
x[n] & \longrightarrow H_{z}(z) \rightarrow y[n] \\
u[n] & \rightarrow v[-n], \quad u[n] \rightarrow H_{z}(z) \rightarrow w[n]
\end{align*}
\]
Zero-Phase Transfer Function

- It is easy to verify the above scheme in the frequency domain.
- Let $X(e^{j\omega})$, $V(e^{j\omega})$, $U(e^{j\omega})$, $W(e^{j\omega})$, and $Y(e^{j\omega})$ denote the DTFTs of $x[n]$, $v[n]$, $u[n]$, $w[n]$, and $y[n]$, respectively.
- From the figure shown earlier and making use of the symmetry relations we arrive at the relations between various DTFTs as given on the next slide.

$H(z)H(z^{-1})^{-1} = \sum_{n=-\infty}^{\infty} x[n]u[n] - \sum_{n=-\infty}^{\infty} v[n]v[n] = \sum_{n=-\infty}^{\infty} w[n]w[n] - \sum_{n=-\infty}^{\infty} y[n]y[n] = \sum_{n=-\infty}^{\infty} x[n]v[n] = \sum_{n=-\infty}^{\infty} u[n]w[n] = \sum_{n=-\infty}^{\infty} y[n]w[n]$

Combining the above equations we get


The function `filtfilt` implements the above zero-phase filtering scheme.

In the case of a causal transfer function with a nonzero phase response, the phase distortion can be avoided by ensuring that the transfer function has a unity magnitude and a linear-phase characteristic in the frequency band of interest.

The most general type of a filter with a linear phase has a frequency response given by

$H(e^{j\omega}) = e^{-j\omega D}$

which has a linear phase from $\omega = 0$ to $\omega = 2\pi$.

Note also $|H(e^{j\omega})| = 1$ and $\tau(\omega) = D$

Linear-Phase Transfer Function

- The output $y[n]$ of this filter to an input $x[n] = A e^{j\omega n}$ is then given by
  
  $y[n] = A e^{-j\omega D} e^{j\omega n} = A e^{j\omega (n-D)}$

- If $x_a(t)$ and $y_a(t)$ represent the continuous-time signals whose sampled versions, sampled at $t = nT$, are $x[n]$ and $y[n]$ given above, then the delay between $x_a(t)$ and $y_a(t)$ is precisely the group delay of amount $D$.

- If $D$ is an integer, then $y[n]$ is identical to $x[n]$, but delayed by $D$ samples.
- If $D$ is not an integer, $y[n]$, being delayed by a fractional part, is not identical to $x[n]$.
- In the latter case, the waveform of the underlying continuous-time output is identical to the waveform of the underlying continuous-time input and delayed $D$ units of time.
Linear-Phase Transfer Function

- If it is desired to pass input signal components in a certain frequency range undistorted in both magnitude and phase, then the transfer function should exhibit a unity magnitude response and a linear-phase response in the band of interest.

- Since the signal components in the stopband are blocked, the phase response in the stopband can be of any shape.

- Example - Determine the impulse response of an ideal lowpass filter with a linear phase response:

$$H_{LP}(e^{j\omega}) = \begin{cases} e^{-jn_0\omega_c}, & 0 < |\omega| < \omega_c \\ 0, & \omega_c \leq |\omega| \leq \pi \end{cases}$$

- Applying the frequency-shifting property of the DTFT to the impulse response of an ideal zero-phase lowpass filter we arrive at

$$h_{LP}[n] = \frac{\sin \omega_c (n - n_0)}{\pi(n - n_0)}, \quad -\infty < n < \infty$$

- As before, the above filter is noncausal and of doubly infinite length, and hence, unrealizable.

- By truncating the impulse response to a finite number of terms, a realizable FIR approximation to the ideal lowpass filter can be developed.

- The truncated approximation may or may not exhibit linear phase, depending on the value of $n_0$ chosen.

- If we choose $n_0 = N/2$ with $N$ a positive integer, the truncated and shifted approximation

$$\hat{h}_{LP}[n] = \frac{\sin \omega_c (n - N/2)}{\pi(n - N/2)}, \quad 0 \leq n \leq N$$

will be a length $N+1$ causal linear-phase FIR filter.
**Linear-Phase Transfer Function**

- Figure below shows the filter coefficients obtained using the function $\text{sinc}$ for two different values of $N$

![Diagram](image)

**Zero-Phase Response**

- Because of the symmetry of the impulse response coefficients as indicated in the two figures, the frequency response of the truncated approximation can be expressed as:

$$\hat{H}_{LP}(e^{j\omega}) = \sum_{n=0}^{N} h^{LP}[n] e^{-j\omega n} = e^{-j\omega N/2} \hat{H}_{LP}(\omega)$$

where $\hat{H}_{LP}(\omega)$, called the zero-phase response or amplitude response, is a real function of $\omega$

**Minimum-Phase and Maximum-Phase Transfer Functions**

- Consider the two 1st-order transfer functions:

$$H_1(z) = \frac{z+b}{z+a}, \quad H_2(z) = \frac{bz+1}{z+a}, \quad |a|<1, \quad |b|<1$$

- Both transfer functions have a pole inside the unit circle at the same location $z = -a$ and are stable

- But the zero of $H_1(z)$ is inside the unit circle at $z = -b$, whereas, the zero of $H_2(z)$ is at $z = -\frac{1}{b}$ situated in a mirror-image symmetry

**Minimum-Phase and Maximum-Phase Transfer Functions**

- However, both transfer functions have an identical magnitude function as

$$H_1(z)H_1(z^{-1}) = H_2(z)H_2(z^{-1})$$

- The corresponding phase functions are

$$\arg[H_1(e^{j\omega})] = \tan^{-1}\frac{\sin\omega}{\omega} - \tan^{-1}\frac{\sin\omega}{\omega}$$

$$\arg[H_2(e^{j\omega})] = \tan^{-1}\frac{b\sin\omega}{1+b\cos\omega} - \tan^{-1}\frac{\sin\omega}{\omega}$$
Minimum-Phase and Maximum-Phase Transfer Functions

- From this figure it follows that $H_2(z)$ has an excess phase lag with respect to $H_1(z)$.

- The excess phase lag property of $H_2(z)$ with respect to $H_1(z)$ can also be explained by observing that we can write

$$H_2(z) = \frac{bz + 1}{z + a} = \left(\frac{bz + 1}{z + a}\right) \left(\frac{z + b}{z + a}\right)$$

where $A(z) = \left(\frac{z + b}{z + a}\right)$ is a stable allpass function.

- The unwrapped phase functions of $H_1(z)$ and $H_2(z)$ are thus related through

$$\arg[H_2(e^{j\omega})] = \arg[H_1(e^{j\omega})] + \arg[A(e^{j\omega})]$$

- As the unwrapped phase function of a stable first-order allpass function is a negative function of $\omega$, it follows from the above that $H_2(z)$ has indeed an excess phase lag with respect to $H_1(z)$.

Minimum-Phase and Maximum-Phase Transfer Functions

- Generalizing the above result, let $H_m(z)$ be a causal stable transfer function with all zeros inside the unit circle and let $H(z)$ be another causal stable transfer function satisfying $|H(e^{j\omega})| = |H_m(e^{j\omega})|$. These two transfer functions are then related through $H(z) = H_m(z)A(z)$ where $A(z)$ is a causal stable allpass function.

Minimum-Phase and Maximum-Phase Transfer Functions

- A causal stable transfer function with all zeros outside the unit circle is called a maximum-phase transfer function.

- A causal stable transfer function with zeros inside and outside the unit circle is called a mixed-phase transfer function.

Minimum-Phase and Maximum-Phase Transfer Functions

- Example – Consider the mixed-phase transfer function

$$H(z) = \frac{2(1 + 0.3z^{-1})(0.4 - z^{-1})}{(1 - 0.2z^{-1})(1 + 0.5z^{-1})}$$

- We can rewrite $H(z)$ as

$$H(z) = \frac{2(1 + 0.3z^{-1})(1 - 0.4z^{-1})}{(1 - 0.2z^{-1})(1 + 0.5z^{-1})} \left[\frac{0.4 - z^{-1}}{1 - 0.4z^{-1}}\right]$$