LTI Discrete-Time Systems in the Transform Domain

• An LTI discrete-time system is completely characterized in the time-domain by its impulse response sequence \{h[n]\}.
• Thus, the transform-domain representation of a discrete-time signal can also be equally applied to the transform-domain representation of an LTI discrete-time system.

LTI Discrete-Time Systems in the Transform Domain

• Such transform-domain representations provide additional insight into the behavior of such systems.
• It is easier to design and implement these systems in the transform-domain for certain applications.
• We consider now the use of the DTFT and the z-transform in developing the transform-domain representations of an LTI system.

Finite-Dimensional LTI Discrete-Time Systems

• In this course we shall be concerned with LTI discrete-time systems characterized by linear constant coefficient difference equations of the form:

\[
\sum_{k=0}^{N} d_k y[n-k] = \sum_{k=0}^{M} p_k x[n-k]
\]

Finite-Dimensional LTI Discrete-Time Systems

• Applying the z-transform to both sides of the difference equation and making use of the linearity and the time-invariance properties of Table 6.2 we arrive at

\[
\sum_{k=0}^{N} d_k z^{-k} Y(z) = \sum_{k=0}^{M} p_k z^{-k} X(z)
\]

where \(Y(z)\) and \(X(z)\) denote the z-transforms of \(y[n]\) and \(x[n]\) with associated ROCs, respectively.

Finite-Dimensional LTI Discrete-Time Systems

• A more convenient form of the z-domain representation of the difference equation is given by

\[
\left(\sum_{k=0}^{N} d_k z^{-k}\right) Y(z) = \left(\sum_{k=0}^{M} p_k z^{-k}\right) X(z)
\]

The Transfer Function

• A generalization of the frequency response function.
• The convolution sum description of an LTI discrete-time system with an impulse response \(h[n]\) is given by

\[
y[n] = \sum_{k=-\infty}^{\infty} h[k] x[n-k]
\]
The Transfer Function

• Taking the $z$-transforms of both sides we get

$$Y(z) = \sum_{n=\infty}^{0} y[n]z^{-n} = \sum_{n=\infty}^{0} \left( \sum_{k=\infty}^{0} h[k]x[n-k] \right) z^{-n}$$

$$= \sum_{k=\infty}^{0} h[k] \left( \sum_{n=\infty}^{0} x[n-k] z^{-n} \right)$$

$$= \sum_{k=\infty}^{0} h[k] \left( \sum_{\ell=\infty}^{0} x[\ell] z^{-(\ell+k)} \right)$$

• Or, equivalently as

$$H(z) = \frac{Y(z)}{X(z)}$$

• Hence, $H(z)$ is the transfer function or the system function

• The inverse $z$-transform of the transfer function $H(z)$ yields the impulse response $h[n]$

• Consider an LTI discrete-time system characterized by a difference equation

$$\sum_{k=\infty}^{0} d[k]y[n-k] = \sum_{k=\infty}^{M} p[k]x[n-k]$$

• Its transfer function is obtained by taking the $z$-transform of both sides of the above equation

$$H(z) = \frac{\sum_{k=\infty}^{M} p[k] z^{-k}}{\sum_{k=\infty}^{N} d[k] z^{-k}}$$

• Or, equivalently as

$$H(z) = \frac{P_0 z^{(N-M)}}{D_0} \prod_{k=1}^{M} \left( z - \xi_k \right) \prod_{k=1}^{N} \left( z - \lambda_k \right)$$

• $\xi_1, \xi_2, \ldots, \xi_M$ are the finite zeros, and $\lambda_1, \lambda_2, \ldots, \lambda_N$ are the finite poles of $H(z)$

• If $N > M$, there are additional $(N - M)$ zeros at $z = 0$

• If $N < M$, there are additional $(M - N)$ poles at $z = 0$
The Transfer Function

• For a causal IIR digital filter, the impulse response is a causal sequence.
• The ROC of the causal transfer function
  \[ H(z) = \frac{b_0}{a_0} z^{(N-M)} \prod_{k=1}^{M} (z - \xi_k) \prod_{k=1}^{N} (z - \lambda_k) \]
  is thus exterior to a circle going through the pole furthest from the origin.
• Thus the ROC is given by \(|z| > \max |\lambda_k|\).

The Transfer Function

• Example - Consider the M-point moving-average FIR filter with an impulse response
  \[ h[n] = \begin{cases} \frac{1}{M}, & 0 \leq n \leq M-1 \\ 0, & \text{otherwise} \end{cases} \]
• Its transfer function is then given by
  \[ H(z) = \frac{1}{M} \sum_{n=0}^{M-1} z^{-n} = \frac{1 - z^{-M}}{M(1 - z^{-1})} = \frac{z^{M-1}}{M(z^{M} - z)} \]

The Transfer Function

• The transfer function has M zeros on the unit circle at \(z = e^{j2\pi k/M}, 0 \leq k \leq M-1\).
• There are M - 1 poles at z = 0 and a single pole at z = 1.
• The ROC is the entire z-plane except z = 0.

The Transfer Function

• Alternate forms:
  \[ H(z) = \frac{z^2 - 1.2z + 1}{z^3 - 1.3z^2 + 1.04z - 0.222} \]
  \[ = \frac{(z - 0.6 + j0.8)(z - 0.6 - j0.8)}{(z - 0.3)(z - 0.5 + j0.7)(z - 0.5 - j0.7)} \]
• Note: Poles farthest from z = 0 have a magnitude \(\sqrt{0.74}\).
• ROC: \(|z| > \sqrt{0.74}\).

Frequency Response from Transfer Function

• If the ROC of the transfer function \(H(z)\) includes the unit circle, then the frequency response \(H(e^{j\omega})\) of the LTI digital filter can be obtained simply as follows:
  \[ H(e^{j\omega}) = H(z) \big|_{z=e^{j\omega}} \]
• For a real coefficient transfer function \(H(z)\), it can be shown that
  \[ |H(e^{j\omega})|^2 = H(e^{j\omega})H^*(e^{j\omega}) = H(z)H(z^{-1}) \big|_{z=e^{j\omega}} \]
Frequency Response from Transfer Function

For a stable rational transfer function in the form
\[ H(z) = \frac{p_0}{d_0} z^{-N+M} \prod_{k=1}^{M} \frac{z - \xi_k}{z - \lambda_k} \]
the factored form of the frequency response is given by
\[ H(e^{j\omega}) = \frac{p_0}{d_0} e^{-j\omega(N-M)} \prod_{k=1}^{M} \frac{e^{j\omega} - \xi_k}{e^{j\omega} - \lambda_k} \]

Frequency Response from Transfer Function

which reduces to
\[ |H(e^{j\omega})| = \left| \frac{p_0}{d_0} \prod_{k=1}^{M} \frac{e^{j\omega} - \xi_k}{e^{j\omega} - \lambda_k} \right| \]

• The phase response for a rational transfer function is of the form
\[ \arg H(e^{j\omega}) = \arg \left( \frac{p_0}{d_0} \right) + \omega(N-M) + \sum_{k=1}^{M} \arg(e^{j\omega} - \xi_k) - \sum_{k=1}^{N} \arg(e^{j\omega} - \lambda_k) \]

Geometric Interpretation of Frequency Response Computation

• The factored form of the frequency response
\[ H(e^{j\omega}) = \frac{p_0}{d_0} e^{-j\omega(N-M)} \prod_{k=1}^{M} \frac{e^{j\omega} - \xi_k}{e^{j\omega} - \lambda_k} \]
is convenient to develop a geometric interpretation of the frequency response computation from the pole-zero plot as \( \omega \) varies from 0 to 2\( \pi \) on the unit circle

• It is convenient to visualize the contributions of the zero factor \( z - \xi_k \) and the pole factor \( z - \lambda_k \) from the factored form of the frequency response
• The magnitude function is given by
\[ |H(e^{j\omega})| = \left| \frac{p_0}{d_0} \prod_{k=1}^{M} \frac{e^{j\omega} - \xi_k}{e^{j\omega} - \lambda_k} \right| \]

• The magnitude-squared function of a real-coefficient transfer function can be computed using
\[ |H(e^{j\omega})|^2 = \left| \frac{p_0^2}{d_0^2} \prod_{k=1}^{M} \frac{(e^{j\omega} - \xi_k)(e^{-j\omega} - \xi_k^*)}{(e^{j\omega} - \lambda_k)(e^{-j\omega} - \lambda_k^*)} \right| \]

• The geometric interpretation can be used to obtain a sketch of the response as a function of the frequency
• A typical factor in the factored form of the frequency response is given by
\[ e^{j\omega} - \rho e^{j\theta} \]
where \( \rho e^{j\theta} \) is a zero if it is zero factor or is a pole if it is a pole factor
Geometric Interpretation of Frequency Response Computation

- As shown below in the $z$-plane the factor $(e^{j\omega} - pe^{j\theta})$ represents a vector starting at the point $z = pe^{j\theta}$ and ending on the unit circle at $z = e^{j\omega}$.

- As $\omega$ is varied from 0 to $2\pi$, the tip of the vector moves counterclockwise from the point $z = 1$ tracing the unit circle and back to the point $z = 1$.

- As indicated by

$$|H(e^{j\omega})| = \left|\frac{p_0}{d_0}\prod_{k=1}^{M} e^{j\omega} - \xi_k\right| \prod_{k=1}^{N} e^{j\omega} - \lambda_k$$

the magnitude response $|H(e^{j\omega})|$ at a specific value of $\omega$ is given by the product of the magnitudes of all zero vectors divided by the product of the magnitudes of all pole vectors.

- Likewise, from

$$\arg H(e^{j\omega}) = \arg(p_0/d_0) + \omega(N - M) + \sum_{k=1}^{M} \arg(e^{j\omega} - \xi_k) - \sum_{k=1}^{N} \arg(e^{j\omega} - \lambda_k)$$

we observe that the phase response at a specific value of $\omega$ is obtained by adding the phase of the term $p_0/d_0$ and the linear-phase term $\omega(N - M)$ to the sum of the angles of the zero vectors minus the angles of the pole vectors.

- Thus, an approximate plot of the magnitude and phase responses of the transfer function of an LTI digital filter can be developed by examining the pole and zero locations.

- Now, a zero (pole) vector has the smallest magnitude when $\omega = \phi$.

- To highly attenuate signal components in a specified frequency range, we need to place zeros very close to or on the unit circle in this range.

- Likewise, to highly emphasize signal components in a specified frequency range, we need to place poles very close to or on the unit circle in this range.
Stability Condition in Terms of the Pole Locations

• A causal LTI digital filter is BIBO stable if and only if its impulse response \( h[n] \) is absolutely summable, i.e.,
  \[ S = \sum_{n=-\infty}^{\infty} |h[n]| < \infty \]

• We now develop a stability condition in terms of the pole locations of the transfer function \( H(z) \)

Stability Condition in Terms of the Pole Locations

• The ROC of the \( z \)-transform \( H(z) \) of the impulse response sequence \( h[n] \) is defined by values of \( |z| = r \) for which \( h[n]r^{-n} \) is absolutely summable

• Thus, if the ROC includes the unit circle \( |z| = 1 \), then the digital filter is stable, and vice versa

Stability Condition in Terms of the Pole Locations

• In addition, for a stable and causal digital filter for which \( h[n] \) is a right-sided sequence, the ROC will include the unit circle and entire \( z \)-plane including the point \( z = \infty \)

• An FIR digital filter with bounded impulse response is always stable

Stability Condition in Terms of the Pole Locations

• Example - Consider the causal IIR transfer function
  \[ H(z) = \frac{1}{1 - 1.845z^{-1} + 0.850586z^{-2}} \]

• The plot of the impulse response coefficients is shown on the next slide

• On the other hand, an IIR filter may be unstable if not designed properly

• In addition, an originally stable IIR filter characterized by infinite precision coefficients may become unstable when coefficients get quantized due to implementation

• As can be seen from the above plot, the impulse response coefficient \( h[n] \) decays rapidly to zero value as \( n \) increases
Stability Condition in Terms of the Pole Locations

• The absolute summability condition of $h[n]$ is satisfied
• Hence, $H(z)$ is a stable transfer function
• Now, consider the case when the transfer function coefficients are rounded to values with 2 digits after the decimal point:

$$\hat{H}(z) = \frac{1}{1 - 1.85z^{-1} + 0.85z^{-2}}$$

A plot of the impulse response of $\hat{h}[n]$ is shown below.

Stability Condition in Terms of the Pole Locations

• In this case, the impulse response coefficient $\hat{h}[n]$ increases rapidly to a constant value as $n$ increases
• Hence, the absolute summability condition of is violated
• Thus, $\hat{H}(z)$ is an unstable transfer function

The stability testing of a IIR transfer function is therefore an important problem
• In most cases it is difficult to compute the infinite sum

$$S = \sum_{n=0}^{\infty} |h[n]| < \infty$$
• For a causal IIR transfer function, the sum $S$ can be computed approximately as

$$S_K = \sum_{n=0}^{K} |h[n]|$$

Stability Condition in Terms of the Pole Locations

• The partial sum is computed for increasing values of $K$ until the difference between a series of consecutive values of $S_K$ is smaller than some arbitrarily chosen small number, which is typically $10^{-6}$
• For a transfer function of very high order this approach may not be satisfactory
• An alternate, easy-to-test, stability condition is developed next

Consider the causal IIR digital filter with a rational transfer function $H(z)$ given by

$$H(z) = \frac{\sum_{k=0}^{M} p_k z^{-k}}{\sum_{k=0}^{N} d_k z^{-k}}$$
• Its impulse response $\{\hat{h}[n]\}$ is a right-sided sequence
• The ROC of $H(z)$ is exterior to a circle going through the pole furthest from $z = 0$
Stability Condition in Terms of the Pole Locations

• But stability requires that \( \{h[n]\} \) be absolutely summable
• This in turn implies that the DTFT \( H(e^{j\omega}) \) of \( \{h[n]\} \) exists
• Now, if the ROC of the z-transform \( H(z) \) includes the unit circle, then
  \[ H(e^{j\omega}) = H(z) \bigg|_{z = e^{j\omega}} \]

Conclusion: All poles of a causal stable transfer function \( H(z) \) must be strictly inside the unit circle

The stability region (shown shaded) in the \( z \)-plane is shown below

Example - The factored form of

\[
H(z) = \frac{1}{1-0.845z^{-1} + 0.850586z^{-2}}
\]

is

\[
H(z) = \frac{1}{(1-0.902z^{-1})(1-0.943z^{-1})}
\]

which has a real pole at \( z = 0.902 \) and a real pole at \( z = 0.943 \)

Since both poles are inside the unit circle, \( H(z) \) is BIBO stable

Example - The factored form of

\[
\hat{H}(z) = \frac{1}{1-1.85z^{-1} + 0.85z^{-2}}
\]

is

\[
\hat{H}(z) = \frac{1}{(1-z^{-1})(1-0.85z^{-1})}
\]

which has a real pole on the unit circle at \( z = 1 \) and the other pole inside the unit circle

Since both poles are not inside the unit circle, \( \hat{H}(z) \) is unstable