Digital Integrators

- An important component in many applications
- Ideal digital integrator frequency response

$$H_{INT}(e^{j\omega}) = \frac{1}{j\omega}$$

- Practical digital integrators are designed to have a frequency response approximating the above expression and are based on numerical integration methods

First-Order IIR Digital Integrators

Forward Rectangular Integrator is based on forward rectangular method of integration

- Its time-domain input-output relation is

$$y[n] = y[n-1] + T \cdot x[n-1]$$

where T is the sampling period

- Its transfer function is given by

$$H_{FR}(z) = T\left(\frac{z^{-1}}{1-z^{-1}}\right)$$

First-Order IIR Digital Integrators

Backward Rectangular Integrator is based on backward rectangular method of integration

- Its time-domain input-output relation is

$$y[n] = y[n-1] + T \cdot x[n]$$

- Its transfer function is given by

$$H_{BR}(z) = T\left(\frac{1}{1-z^{-1}}\right)$$

- Note:

$$\left|H_{FR}(e^{j\omega})\right| = \left|H_{BR}(e^{j\omega})\right|$$

First-Order IIR Digital Integrators

Trapezoidal Integrator is based on the trapezoidal method of integration

- Its time-domain input-output relation is

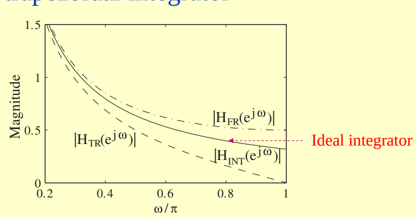
$$y[n] = y[n-1] + \frac{T}{2} \cdot (x[n] + x[n-1])$$

- Its transfer function is given by

$$H_{TR}(z) = \frac{T}{2}\left(\frac{1+z^{-1}}{1-z^{-1}}\right)$$

First-Order IIR Digital Integrators

- The magnitude response of the ideal integrator is between those of the rectangular integrator and the trapezoidal integrator



Second-Order IIR Digital Integrators

Simpson integrator is based on the Simpson method of integration and provides an improved numerical result

- Its time-domain input-output relation is

$$y[n] = y[n-2] + \frac{T}{3} \cdot (x[n] + 4x[n-1] + x[n-2])$$

- Its transfer function is given by

$$H_{SI}(z) = \frac{T}{3}\left(\frac{1+4z^{-1}+z^{-2}}{1-z^{-2}}\right)$$

Digital Differentiators

- Employed to perform the differentiation operation on the discrete-time version of a continuous-time signal
- Frequency response of an ideal discrete-time differentiator is given by

$$H(e^{j\omega}) = j\omega \quad \text{for } 0 \le |\omega \le \pi|$$

which has a linear magnitude response from dc to $\omega = \pi$

Digital Differentiators

- A practical discrete-time differentiator is used to perform the differentiation operation in the low frequency range and is thus designed to have a linear magnitude response from dc to a frequency smaller than p

Simple FIR Digital Differentiators

First-Difference Differentiator is a first-order FIR discrete-time system with a time-domain input-output relation given by

$$y[n] = x[n] - x[n-1]$$

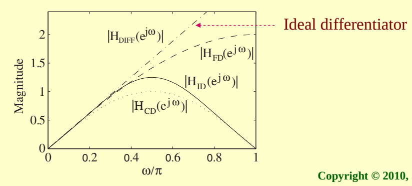
- Its transfer function is given by

$$H_{FD}(z) = 1 - z^{-1}$$

which is same as that of a first-order FIR highpass filter described earlier

Simple FIR Digital Differentiators

- Main drawback of the first-difference differentiator is that it also amplifies the high frequency noise often present in many signals



Simple FIR Digital Differentiators

Central-Difference Differentiator avoids the noise amplification problem of the first-difference differentiator

- Its time-domain input-output relation is

$$y[n] = \tfrac{1}{2}(x[n] - x[n-2])$$

- Its transfer function is given by

$$H_{CD}(z) = \tfrac{1}{2}(1 - z^{-2})$$

- It has a linear magnitude response in a very small low-frequency range

Higher-Order FIR Digital Differentiator

- The time-domain input-output relation of a higher-order FIR digital differentiator is given by

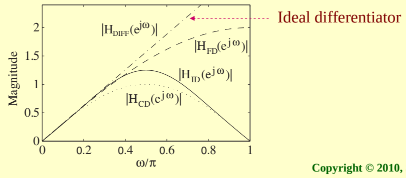
$$y[n] = -\tfrac{1}{16}x[n] + x[n-2] - x[n-4] + \tfrac{1}{16}x[n-6]$$

- Its transfer function is given by

$$H_{ID}(z) = -\tfrac{1}{16} + z^{-2} - z^{-4} + \tfrac{1}{16}z^{-6}$$

Higher-Order FIR Digital Differentiator

- Its magnitude response, scaled by a factor 0.6 is shown below
- The frequency range of operation of this differentiator is from dc to $\omega = 0.34\pi$



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DC Blockers

- In some applications it is necessary to remove the dc bias present in a signal before other signal processing algorithms can be applied
- The ideal dc blocker has an infinite attenuation at dc ($\omega = 0$) and passes all input signals with non-zero frequencies

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DC Blockers

- As a result, a dc blocker is essentially a highpass filter with a transfer function having at least one zero at $z = 1$ and unity magnitude response for $\omega \neq 0$
- We describe next several simple FIR and IIR filters that can be used as dc blockers

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DC Blockers

Simple FIR DC Blocker

- The simple first-order FIR differentiator

$$H_{FD}(z) = 1 - z^{-1}$$

has a zero at $z = 1$ and thus blocks the dc component of a signal quite well

- However, very low-frequency spectral components close to $\omega = 0$ are also attenuated as can be seen from its magnitude response plot in Slide No. 13

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DC Blockers

Simple IIR DC Blocker

- To boost the dropping magnitude of the simple FIR dc blocker $H_{FD}(z)$ near dc, one solution is to cascade it with an all-pole first-order IIR filter with a transfer function

$$G(z) = \frac{1}{1 - \alpha z^{-1}}$$

where α is real and $0 < \alpha < 1$

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DC Blockers

- The IIR filter $G(z)$ is often called a leaky integrator
- The magnitude response of the nonlinear-phase cascaded differentiator/integrator

$$H(z) = G(z)H_{FD}(z) = \frac{1 - z^{-1}}{1 - \alpha z^{-1}}$$

is shown in the next page for various values of α

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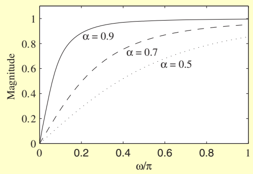
DC Blockers



- Note: The transfer function $F(z)$ is the same as that of the first-order IIR highpass filter $H_{HP}(z)$ except for the scale factor $(1+\alpha)/2$

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DC Blockers

Higher-Order FIR DC Blocker

- A linear-phase dc blocker can be implemented by a delay-complementary Type 1 moving average filter
- A recursive form of the transfer function of an M-point (M odd) moving average filter is given by

$$H_{MA}(z) = \frac{1}{M}\left(\frac{1-z^{-M}}{1-z^{-1}}\right)$$

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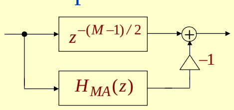
DC Blockers

- The transfer function of the linear-phase dc blocker is thus given by

$$F(z) = z^{-(M-1)/2} - H_{MA}(z)$$

- Its schematic representation is shown below



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DC Blockers

- If M is a power-of-2 integer, then the scaling factor $\frac{1}{M}$ can be implemented using binary shift-and-add operations, avoiding the multiplication operation
- However, in this case the delay unit $z^{-(M-1)/2}$ develops a fractional delay making it difficult to synchronize the two sequences at the output of the adder

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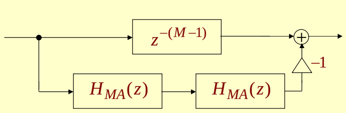
DC Blockers

- One way to avoid this problem is to form the delay-complementary of the cascade of two moving average filters as indicated below



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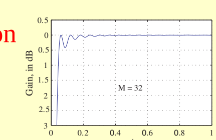
DC Blockers

- The modified structure requires a delay unit $z^{-(M-1)}$ and provides an integer-valued delay
- Gain response of this dc blocker for $M = 32$ is shown below
- It has an infinite attenuation at dc and a peak passband ripple of about 0.42 dB



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Comb Filters

- The simple filters discussed so far are characterized either by a single passband and/or a single stopband
- There are applications where filters with multiple passbands and stopbands are required
- The **comb filter** is an example of such filters

Comb Filters

- In its most general form, a comb filter has a frequency response that is a periodic function of ω with a period $2\pi/L$, where L is a positive integer
- If $H(z)$ is a filter with a single passband and/or a single stopband, a comb filter can be easily generated from it by replacing each delay in its realization with L delays resulting in a structure with a transfer function given by $G(z) = H(z^L)$

Comb Filters

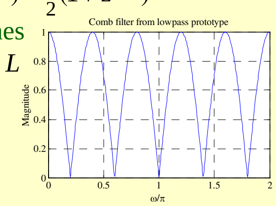
- If $|H(e^{j\omega})|$ exhibits a peak at ω_p, then $|G(e^{j\omega})|$ will exhibit L peaks at $\omega_p k/L$, $0 \le k \le L-1$ in the frequency range $0 \le \omega < 2\pi$
- Likewise, if $|H(e^{j\omega})|$ has a notch at ω_o, then $|G(e^{j\omega})|$ will have L notches at $\omega_o k/L$, $0 \le k \le L-1$ in the frequency range $0 \le \omega < 2\pi$
- A comb filter can be generated from either an FIR or an IIR prototype filter

FIR Comb Filters

- For example, the comb filter generated from the prototype lowpass FIR filter $H_0(z) = \frac{1}{2}(1+z^{-1})$ has a transfer function

$$G_0(z) = H_0(z^L) = \tfrac{1}{2}(1+z^{-L})$$

- $|G_0(e^{j\omega})|$ has L notches at $\omega = (2k+1)\pi/L$ and L peaks at $\omega = 2\pi\, k/L$, $0 \le k \le L-1$, in the frequency range $0 \le \omega < 2\pi$

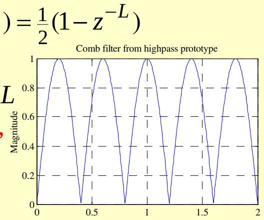


FIR Comb Filters

- For example, the comb filter generated from the prototype highpass FIR filter $H_1(z) = \frac{1}{2}(1-z^{-1})$ has a transfer function

$$G_1(z) = H_1(z^L) = \tfrac{1}{2}(1-z^{-L})$$

- $|G_1(e^{j\omega})|$ has L peaks at $\omega = (2k+1)\pi/L$ and L notches at $\omega = 2\pi\, k/L$, $0 \le k \le L-1$, in the frequency range $0 \le \omega < 2\pi$



FIR Comb Filters

- Depending on applications, comb filters with other types of periodic magnitude responses can be easily generated by appropriately choosing the prototype filter
- For example, the M-point moving average filter

$$H(z) = \frac{1-z^{-M}}{M(1-z^{-1})}$$

has been used as a prototype

FIR Comb Filters

- This filter has a peak magnitude at $\omega = 0$, and $M-1$ notches at $\omega = 2\pi\ell/M, 1 \le \ell \le M-1$
- The corresponding comb filter has a transfer function

$$G(z) = \frac{1-z^{-LM}}{M(1-z^{-L})}$$

whose magnitude has L peaks at $\omega = 2\pi k/L$, $0 \le k \le L-1$ and $L(M-1)$ notches at $\omega = 2\pi k/LM, 1 \le k \le L(M-1)$

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IIR Comb Filters

- The transfer functions of the simplest forms of the prototype IIR filter are given by

$$H_0(z) = K\frac{1-z^{-1}}{1-\alpha z^{-1}}, \quad H_1(z) = K\frac{1+z^{-1}}{1-\alpha z^{-1}}$$

where $|\alpha| < 1$ for stability

- Note: $H_0(z)$ is a highpass filter with a zero at $z = 1$ and $H_1(z)$ is a lowpass filter with a zero at $z = -1$

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IIR Comb Filters

- For a maximum gain of 0 dB, the scale factor K of $H_0(z)$ should be set equal to $(1+\alpha)/2$ and the scale factor K of $H_1(z)$ should be set equal to $(1-\alpha)/2$
- The corresponding transfer functions of the comb filters of order L are

$$G_0(z) = K\frac{1-z^{-L}}{1-\alpha z^{-L}}, \quad G_1(z) = K\frac{1+z^{-L}}{1-\alpha z^{-L}}$$

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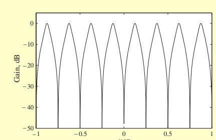
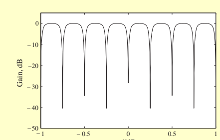
IIR Comb Filters

- Gain responses of the IIR comb filters generated from $H_0(z)$ and $H_1(z)$ for $L = 8$ are shown below



$G_0(z) = H_0(z^8)$



$G_1(z) = H_1(z^8)$

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