Realization of Allpass Filters

• An $M$-th order real-coefficient allpass transfer function $A_M(z)$ is characterized by $M$ unique coefficients as here the numerator is the mirror-image polynomial of the denominator.
• A direct form realization of $A_M(z)$ requires $2M$ multipliers.
• Objective - Develop realizations of $A_M(z)$ requiring only $M$ multipliers.

Realization Using Multiplier Extraction Approach

• Now, an arbitrary allpass transfer function can be expressed as a product of 2nd-order and/or 1st-order allpass transfer functions.
• We consider first the minimum multiplier realization of a 1st-order and a 2nd-order allpass transfer functions.

First-Order Allpass Structures

• Consider first the 1st-order allpass transfer function given by
  $$A_1(z) = \frac{d_1 + z^{-1}}{1 + d_1 z^{-2}}$$
• We shall realize the above transfer function in the form a structure containing a single multiplier $d_1$ as shown below.

First-Order Allpass Structures

- Type 1A:
  $$t_{11} = z^{-1}$$
  $$t_{22} = z^{-1}$$
  $$t_{12} = 1 - z^{-2}$$
- Type 1B:
  $$t_{11} = z^{-1}$$
  $$t_{22} = z^{-1}$$
  $$t_{21} = 1 + z^{-1}$$

First-Order Allpass Structures

- Type 1A:
  $$\begin{align*}
  t_{11} &= z^{-1}, \\
  t_{22} &= z^{-1}, \\
  t_{12} &= 1 - z^{-2}, \\
  t_{21} &= 1 - z^{-2}
  \end{align*}$$
- Type 1B:
  $$\begin{align*}
  t_{11} &= z^{-1}, \\
  t_{22} &= z^{-1}, \\
  t_{12} &= 1 - z^{-1}, \\
  t_{21} &= 1 + z^{-1}
  \end{align*}$$

First-Order Allpass Structures

- Type 1A:
  $$\begin{align*}
  t_{11} &= z^{-1}, \\
  t_{22} &= z^{-1}, \\
  t_{12} &= 1, \\
  t_{21} &= 1 - z^{-2}
  \end{align*}$$
- Type 1B:
  $$\begin{align*}
  t_{11} &= z^{-1}, \\
  t_{22} &= z^{-1}, \\
  t_{12} &= 1 - z^{-1}, \\
  t_{21} &= 1 + z^{-1}
  \end{align*}$$

We now develop the two-pair structure for the Type 1A allpass transfer function.
First-Order Allpass Structures

- From the transfer parameters of this allpass we arrive at the input-output relations:
  \[ Y_2 = X_1 - z^{-1}X_2 \]
  \[ Y_1 = z^{-1}X_1 + (1 - z^{-2})X_2 = z^{-1}Y_2 + X_2 \]
- A realization of the above two-pair is sketched below

Second-Order Allpass Structures

- By constraining the \( X_2, Y_2 \) terminal-pair with the multiplier \( d_1 \), we arrive at the Type 1A allpass filter structure shown below

Type 3 Allpass Structures

- Type 3 allpass transfer function:
  \[ A_3(z) = \frac{d_2 + d_1 z^{-1} + z^{-2}}{1 + d_1 z^{-1} + d_2 z^{-2}} \]
Type 3 Allpass Structures

Realization Using Multiplier Extraction Approach

- Example - Realize
  \[ A_3(z) = \frac{-0.2 + 0.18z^{-1} + 0.4z^{-2} + z^{-3}}{1 + 0.4z^{-1} + 0.18z^{-2} - 0.2z^{-3}} \]
  \[ = \frac{(-0.4 + z^{-1})(0.5 + 0.8z^{-1} + 0.5z^{-2})}{(1 - 0.4z^{-1})(1 + 0.8z^{-1} + 0.5z^{-2})} \]

- A 3-multiplier cascade realization of the above allpass transfer function is shown below

Realization Using Two-Pair Extraction Approach

- Let
  \[ A_m(z) = \frac{d_m + d_m' z^{-1} + \cdots + d_{m-1}' z^{-(m-1)} + z^{-m}}{1 + d_1 z^{-1} + \cdots + d_{m-2} z^{-(m-2)} + d_m z^{-m}} \]
- We use the recursion
  \[ A_{m-1}(z) = \left( \frac{A_m(z) - k_m}{1 - k_m A_m(z)} \right), \ m = M, M-1, \ldots, 1 \]
  where \( k_m = A_m(\infty) = d_m \)
- It has been shown earlier that \( A_M(z) \) is stable if and only if
  \[ k_m^2 < 1 \quad \text{for} \ m = M, M-1, \ldots, 1 \]

Realization Using Two-Pair Extraction Approach

- If the allpass transfer function \( A_{m-1}(z) \) is expressed in the form
  \[ A_{m-1}(z) = \frac{d_{m-1} + d_{m-1}' z^{-1} + \cdots + d_{m-2}' z^{-(m-2)} + z^{-(m-1)}}{1 + d_1 z^{-1} + \cdots + d_{m-2} z^{-(m-2)} + d_m z^{-m}} \]
  then the coefficients of \( A_{m-1}(z) \) are simply related to the coefficients of \( A_m(z) \) through
  \[ d_i = d_i - d_m d_{m-i}, \quad 1 \leq i \leq m - 1 \]
Realization Using Two-Pair Extraction Approach

• The transfer function $\mathcal{A}_m(z) = Y_1/X_1$ of the constrained two-pair can be expressed as
  
  $$\mathcal{A}_m(z) = \frac{t_{11} - (t_1t_2 + t_2t_1) \mathcal{A}_{m-1}(z)}{1 - t_{12} \mathcal{A}_{m-1}(z)}$$

• Comparing the above with

  $$\frac{t_{11} - (t_1t_2 + t_2t_1) \mathcal{A}_{m-1}(z)}{1 - t_{12} \mathcal{A}_{m-1}(z)} = \frac{k_m}{1 + k_m} \mathcal{A}_{m-1}(z)$$

  we arrive at the two-pair transfer parameters

Realization Using Two-Pair Extraction Approach

• Some possible solutions are given below:
  
  $t_{11} = k_m$, $t_{22} = -k_mz^{-1}$, $t_{12} = z^{-1}$, $t_{21} = 1 - k_m$
  
  $t_{11} = k_m$, $t_{22} = -k_mz^{-1}$, $t_{12} = (1 - k_m)z^{-1}$, $t_{21} = 1 + k_m$
  
  $t_{11} = k_m$, $t_{22} = -k_mz^{-1}$, $t_{12} = \sqrt{1 - k_m^2}z^{-1}$, $t_{21} = \sqrt{1 - k_m^2}$
  
  $t_{11} = k_m$, $t_{22} = -k_mz^{-1}$, $t_{12} = (1 - k_m^2)z^{-1}$, $t_{21} = 1$

Realization Using Two-Pair Extraction Approach

• Consider the solution
  
  $t_{11} = k_m$, $t_{22} = -k_mz^{-1}$, $t_{12} = (1 - k_m^2)z^{-1}$, $t_{21} = 1$

  • Corresponding input-output relations are
  
  $$Y_1 = k_mX_1 + (1 - k_m^2)z^{-1}X_2$$
  
  $$Y_2 = X_1 - k_mz^{-1}X_2$$

  • A direct realization of the above equations leads to the 3-multiplier two-pair shown on the next slide

Realization Using Two-Pair Extraction Approach

• Likewise, the transfer parameters
  
  $t_{11} = k_m$, $t_{22} = -k_mz^{-1}$, $t_{12} = \sqrt{1 - k_m^2}z^{-1}$, $t_{21} = \sqrt{1 - k_m^2}$

  lead to the 4-multiplier two-pair structure shown below
Realization Using Two-Pair Extraction Approach

- A 2-multiplier realization can be derived by manipulating the input-output relations:
  \[ Y_1 = k_m X_1 + (1 - k_m^2) z^{-1} X_2 \]
  \[ Y_2 = X_1 - k_m z^{-1} X_2 \]
- Making use of the second equation, we can rewrite the first equation as
  \[ Y_1 = k_m Y_2 + z^{-1} X_2 \]

Realization Using Two-Pair Extraction Approach

- A direct realization of
  \[ Y_1 = k_m Y_2 + z^{-1} X_2 \]
  \[ Y_2 = X_1 - k_m z^{-1} X_2 \]
lead to the 2-multiplier two-pair structure, known as the lattice structure, shown below.

Realization Using Two-Pair Extraction Approach

- Consider the two-pair described by
  \[ t_{11} = k_m, \quad t_{22} = -k_m z^{-1}, \quad t_{12} = (1 - k_m) z^{-1}, \quad t_{21} = 1 + k_m \]
- Its input-output relations are given by
  \[ Y_1 = k_m X_1 + (1 - k_m) z^{-1} X_2 \]
  \[ Y_2 = (1 + k_m) X_1 - k_m z^{-1} X_2 \]
- Define
  \[ V_1 = k_m (X_1 - z^{-1} X_2) \]

Realization Using Two-Pair Extraction Approach

- We can then rewrite the input-output relations as
  \[ Y_1 = V_1 + z^{-1} X_2 \]
  \[ Y_2 = X_1 + V_1 \]
- The corresponding 1-multiplier realization is shown below.

Realization Using Two-Pair Extraction Approach

- An \( m \)-th order allpass transfer function \( A_m(z) \) is then realized by constraining any one of the two-pairs developed earlier by the \( (m-1) \)-th order allpass transfer function \( A_{m-1}(z) \)

Realization Using Two-Pair Extraction Approach

- The process is repeated until the constraining transfer function is \( A_0(z) = 1 \)
- The complete realization of \( A_M(z) \) based on the extraction of the two-pair lattice is shown below.
Realization Using Two-Pair Extraction Approach

• It follows from our earlier discussion that \( \mathcal{A}_M(z) \) is stable if the magnitudes of all multiplier coefficients in the realization are less than 1, i.e., \( |k_m| < 1 \) for \( m = M, M-1, \ldots, 1 \)
• The cascaded lattice allpass filter structure requires \( 2M \) multipliers
• A realization with \( M \) multipliers is obtained if instead the single multiplier two-pair is used

\[
\mathcal{A}_M(z) = \frac{d_3 + d_2 z^{-1} + d_1 z^{-2} + z^{-3}}{1 + d_1 z^{-1} + d_2 z^{-2} + d_3 z^{-3}}
\]

Realization Using Two-Pair Extraction Approach

• Example - Realize \( \mathcal{A}_3(z) \) in the form of a lattice two-pair characterized by the multiplier coefficient \( k_3 = d_3 = -0.2 \) and constrained by a 2nd-order allpass \( \mathcal{A}_2(z) \) as indicated below

\[
\mathcal{A}_3(z) = -0.2 + 0.18 z^{-1} + 0.4 z^{-2} + z^{-3}
\]

\[
1 + 0.4 z^{-1} + 0.18 z^{-2} - 0.2 z^{-3}
\]

Realization Using Two-Pair Extraction Approach

• We first realize \( \mathcal{A}_3(z) \) in the form of a lattice two-pair characterized by the multiplier coefficient \( k_3 = d_3 = -0.2 \) and constrained by a 2nd-order allpass \( \mathcal{A}_2(z) \) as indicated below

\[
\mathcal{A}_3(z) = \frac{d_3' + d_2' z^{-1} + z^{-2}}{1 + d_1' z^{-1} + d_2' z^{-2}}
\]

Realization Using Two-Pair Extraction Approach

• Next, the allpass \( \mathcal{A}_2(z) \) is realized as a lattice two-pair characterized by the multiplier coefficient \( k_2 = d_2 = 0.2708333 \) and constrained by a 2nd-order allpass \( \mathcal{A}_1(z) \) as indicated below

\[
\mathcal{A}_3(z) = \frac{d_3 + d_2 z^{-1} + d_1 z^{-2} + z^{-3}}{1 + d_1 z^{-1} + d_2 z^{-2} + d_3 z^{-3}}
\]

\[
\mathcal{A}_2(z) = \frac{d_3' + d_2' z^{-1} + z^{-2}}{1 + d_1' z^{-1} + d_2' z^{-2}}
\]

\[
\mathcal{A}_1(z) = \frac{d_1' + z^{-1}}{1 + d_1' z^{-1}}
\]

\[
d_1' = \frac{d_1 - d_2 d_3'}{1 - d_2' d_3} = \frac{0.4 - 0.2(0.18)}{1 - (0.2)^2} = 0.4541667
\]

\[
d_2' = \frac{d_2 - d_1 d_3'}{1 - d_2'} = \frac{0.18 - (0.2)(0.4)}{1 - (0.2)^2} = 0.2708333
\]

\[
d_3' = \frac{d_3 - d_2 d_1'}{1 - d_3'} = \frac{d_3'}{1 + d_2'} = \frac{0.2708333}{1 + 0.2708333} = 0.3573771
\]
Realization Using Two-Pair Extraction Approach

- Finally, the allpass \( A_1(z) \) is realized as a lattice two-pair characterized by the multiplier coefficient \( k_1 = d_1 = 0.3573771 \) and constrained by an allpass \( A_0(z) = 1 \) as indicated below:

Cascaded Lattice Realization Using MATLAB

- The M-file `poly2rc` can be used to realize an allpass transfer function in the cascaded lattice form.
- To this end Program 8.4 can be employed.