Least Integral-Squared Error Design of FIR Filters

• Let \( H_d(e^{j\omega}) \) denote the desired frequency response
• Since \( H_d(e^{j\omega}) \) is a periodic function of \( \omega \) with a period \( 2\pi \), it can be expressed as a Fourier series
\[
H_d(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h_d[n] e^{-j\omega n}
\]
where
\[
h_d[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n} d\omega, \quad -\infty \leq n \leq \infty
\]

Least Integral-Squared Error Design of FIR Filters

• In general, \( H_d(e^{j\omega}) \) is piecewise constant with sharp transitions between bands
• In which case, \( \{h_d[n]\} \) is of infinite length and noncausal
• Objective - Find a finite-duration \( \{h_i[n]\} \) of length \( 2M+1 \) whose DTFT \( H_i(e^{j\omega}) \) approximates the desired DTFT \( H_d(e^{j\omega}) \) in some sense

• Commonly used approximation criterion - Minimize the integral-squared error
\[
\Phi = \frac{1}{2\pi} \int_{-\pi}^{\pi} [H_i(e^{j\omega}) - H_d(e^{j\omega})]^2 d\omega
\]
where
\[
H_i(e^{j\omega}) = \sum_{n=-M}^{M} h_i[n] e^{-j\omega n}
\]

Least Integral-Squared Error Design of FIR Filters

• Using Parseval’s relation we can write
\[
\Phi = \sum_{n=-\infty}^{\infty} |h_i[n] - h_d[n]|^2
= \sum_{n=-M}^{M} |h_i[n] - h_d[n]|^2 + \sum_{n=M+1}^{\infty} h_i^2[n] + \sum_{n=-M}^{-1} h_d^2[n]
\]

• It follows from the above that \( \Phi \) is minimum when \( h_i[n] = h_d[n] \) for \( -M \leq n \leq M \)
• \( \Rightarrow \) Best finite-length approximation to ideal infinite-length impulse response in the mean-square sense is obtained by truncation

Least Integral-Squared Error Design of FIR Filters

• A causal FIR filter with an impulse response \( h[n] \) can be derived from \( h_i[n] \) by delaying:
\[
h[n] = h_i[n - M]
\]
• The causal FIR filter \( h[n] \) has the same magnitude response as \( h_i[n] \) and its phase response has a linear phase shift of \( \omega M \) radians with respect to that of \( h_i[n] \)

Impulse Responses of Ideal Filters

• Ideal lowpass filter -
\[
h_{LP}[n] = \frac{\sin \omega_n n}{\pi n}, \quad -\infty \leq n \leq \infty
\]
• Ideal highpass filter -
\[
h_{HP}[n] = \begin{cases} 1 - \frac{\omega_n}{\pi}, & n = 0 \\ \frac{\sin(\omega_n n)}{\pi n}, & n \neq 0 \end{cases}
\]
Impulse Responses of Ideal Filters

• Ideal bandpass filter -

\[ h_{BP}[n] = \begin{cases} \frac{\sin(\omega_2 n)}{\pi n} - \frac{\sin(\omega_1 n)}{\pi n}, & n \neq 0 \\ \frac{\omega_2}{\pi} - \frac{\omega_1}{\pi}, & n = 0 \end{cases} \]

• Ideal bandstop filter -

\[ h_{BS}[n] = \begin{cases} 1 - \frac{(\omega_2 - \omega_1)}{\pi n}, & n = 0 \\ \frac{\sin(\omega_1 n)}{\pi n} - \frac{\sin(\omega_2 n)}{\pi n}, & n \neq 0 \end{cases} \]

• Ideal multiband filter -

\[ H_{ML}(e^{j\omega}) = A_k, \quad \omega_{k-1} \leq \omega \leq \omega_k, \quad k = 1, 2, \ldots, L \]

\[ h_{ML}[n] = \sum_{\ell=1}^{L} (A_\ell - A_{\ell+1}) \frac{\sin(\omega_\ell n)}{\pi n} \]

• Ideal discrete-time Hilbert transformer -

\[ H_{HT}(e^{j\omega}) = \begin{cases} j, & -\pi < \omega < 0 \\ -j, & 0 < \omega < \pi \end{cases} \]

\[ h_{HT}[n] = \begin{cases} 0, & \text{for } n \text{ even} \\ \frac{1}{2\pi n}, & \text{for } n \text{ odd} \end{cases} \]

Gibbs Phenomenon

• Gibbs phenomenon - Oscillatory behavior in the magnitude responses of causal FIR filters obtained by truncating the impulse response coefficients of ideal filters.
Gibbs Phenomenon

- As can be seen, as the length of the lowpass filter is increased, the number of ripples in both passband and stopband increases, with a corresponding decrease in the ripple widths.
- Height of the largest ripples remain the same independent of length.
- Similar oscillatory behavior observed in the magnitude responses of the truncated versions of other types of ideal filters.

Gibbs Phenomenon

- Gibbs phenomenon can be explained by treating the truncation operation as a windowing operation:
  \[ h[n] = h_d[n] \cdot w[n] \]
- In the frequency domain
  \[ H(e^{j\omega}) = \frac{1}{2\pi} \int H_d(e^{j\omega})\Psi(e^{j(\omega-\phi)})d\phi \]
- where \( H_d(e^{j\omega}) \) and \( \Psi(e^{j\omega}) \) are the DTFTs of \( h[n] \) and \( w[n] \), respectively.

Gibbs Phenomenon

- If \( \Psi(e^{j\omega}) \) is a very narrow pulse centered at \( \omega = 0 \) (ideally a delta function) compared to variations in \( H_d(e^{j\omega}) \), then \( H_d(e^{j\omega}) \) will approximate \( H_d(e^{j\omega}) \) very closely.
- Length \( 2M+1 \) of \( w[n] \) should be very large.
- On the other hand, length \( 2M+1 \) of \( h[n] \) should be as small as possible to reduce computational complexity.

Gibbs Phenomenon

- A rectangular window is used to achieve simple truncation:
  \[ w_R[n] = \begin{cases} 1, & 0 \leq |n| \leq M \\ 0, & \text{otherwise} \end{cases} \]
- Presence of oscillatory behavior in \( H(e^{j\omega}) \) is basically due to:
  - 1) \( h_d[n] \) is infinitely long and not absolutely summable, and hence filter is unstable.
  - 2) Rectangular window has an abrupt transition to zero.

Gibbs Phenomenon

- Oscillatory behavior can be explained by examining the DTFT \( \Psi_R(e^{j\omega}) \) of \( w_R[n] \):
  - \( \Psi_R(e^{j\omega}) \) has a main lobe centered at \( \omega = 0 \).
  - Other ripples are called sidelobes.
Gibbs Phenomenon

- Main lobe of \( \Psi_r(e^{j\omega}) \) characterized by its width \( 4\pi/(2M + 1) \) defined by first zero crossings on both sides of \( \omega = 0 \)
- As \( M \) increases, width of main lobe decreases as desired
- Area under each lobe remains constant while width of each lobe decreases with an increase in \( M \)
- Ripples in \( H(e^{j\omega}) \) around the point of discontinuity occur more closely but with no decrease in amplitude as \( M \) increases

Gibbs Phenomenon

- Rectangular window has an abrupt transition to zero outside the range \(-M \leq n \leq M\), which results in Gibbs phenomenon in \( H(e^{j\omega}) \)
- Gibbs phenomenon can be reduced either:
  1. Using a window that tapers smoothly to zero at each end, or
  2. Providing a smooth transition from passband to stopband in the magnitude specifications