Transform-Domain Representation of Discrete-Time Signals

- Three useful representations of discrete-time sequences in the transform domain:
  - Discrete-time Fourier Transform
  - Discrete Fourier Transform
  - $z$ Transform

Discrete-Time Fourier Transform

- Definition - The discrete-time Fourier transform (DTFT) $X(e^{j\omega})$ of a sequence $x[n]$ is given by
  $$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$
- In general, $X(e^{j\omega})$ is a complex function of the real variable $\omega$ and can be written as
  $$X(e^{j\omega}) = X_{re}(e^{j\omega}) + jX_{im}(e^{j\omega})$$

Discrete-Time Fourier Transform

- $X_{re}(e^{j\omega})$ and $X_{im}(e^{j\omega})$ are, respectively, the real and imaginary parts of $X(e^{j\omega})$, and are real functions of $\omega$
- $X(e^{j\omega})$ can alternately be expressed as
  $$X(e^{j\omega}) = |X(e^{j\omega})|e^{j\theta(\omega)}$$
  where
  $$\theta(\omega) = \arg \{X(e^{j\omega})\}$$

Discrete-Time Fourier Transform

- For a real sequence $x[n]$, $|X(e^{j\omega})|$ and $X_{re}(e^{j\omega})$ are even functions of $\omega$, whereas, $\theta(\omega)$ and $X_{im}(e^{j\omega})$ are odd functions of $\omega$
- Note: $X(e^{j\omega}) = |X(e^{j\omega})|e^{j\theta(\omega+2\pi k)}$
- for any integer $k$
- The phase function $\theta(\omega)$ cannot be uniquely specified for any DTFT

Discrete-Time Fourier Transform

- Unless otherwise stated, we shall assume that the phase function $\theta(\omega)$ is restricted to the following range of values:
  $$-\pi \leq \theta(\omega) < \pi$$
  called the principal value
Discrete-Time Fourier Transform

• The DTFTs of some sequences exhibit discontinuities of $2\pi$ in their phase responses
• An alternate type of phase function that is a continuous function of $\omega$ is often used
• It is derived from the original phase function by removing the discontinuities of $2\pi$

Discrete-Time Fourier Transform

• The process of removing the discontinuities is called “unwrapping”
• The continuous phase function generated by unwrapping is denoted as $\theta_c(\omega)$
• In some cases, discontinuities of $\pi$ may be present after unwrapping

Example - The DTFT of the unit sample sequence $\delta[n]$ is given by
$$\Delta(\omega) = \sum_{n=-\infty}^{\infty} \delta[n] e^{-j\omega n} = \delta[0] = 1$$

Example - Consider the causal sequence
$$x[n] = \alpha^n \mu[n], \quad |\alpha| < 1$$

The DTFT is given by
$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \alpha^n \mu[n] e^{-j\omega n} = \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n}$$
$$= \frac{1}{1-\alpha e^{-j\omega}}$$
as $|\alpha e^{-j\omega}| = |\alpha| < 1$

The magnitude and phase of the DTFT $X(e^{j\omega}) = 1/(1 - 0.5e^{-j\omega})$ are shown below

The DTFT $X(e^{j\omega})$ of a sequence $x[n]$ is a continuous function of $\omega$
• It is also a periodic function of $\omega$ with a period $2\pi$:
$$X(e^{j(\omega + 2\pi n)}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j(\omega + 2\pi n)n}$$
$$= \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} e^{-j2\pi kn} = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} = X(e^{j\omega})$$
Discrete-Time Fourier Transform

- Therefore
  \[ X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \]
  represents the Fourier series representation of the periodic function
- As a result, the Fourier coefficients \( x[n] \) can be computed from
  \[ X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \]
  using the Fourier integral
  \[ x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega \]

Discrete-Time Fourier Transform

- Inverse discrete Fourier transform: 
  \[ x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega \]
- Proof:
  \[ x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left( \sum_{\ell=-\infty}^{\infty} x[\ell]e^{-j\omega \ell} \right)e^{j\omega n}d\omega \]

Discrete-Time Fourier Transform

- The order of integration and summation can be interchanged if the summation inside the brackets converges uniformly, i.e., \( X(e^{j\omega}) \) exists
- Then
  \[ \frac{1}{2\pi} \int_{-\pi}^{\pi} \left( \sum_{\ell=-\infty}^{\infty} x[\ell]e^{-j\omega \ell} \right)e^{j\omega n}d\omega \]
  \[ = \sum_{\ell=-\infty}^{\infty} x[\ell] \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j\omega(n-\ell)}d\omega \right) \]
  \[ = \sum_{\ell=-\infty}^{\infty} x[\ell] \frac{\sin \pi(n-\ell)}{\pi(n-\ell)} \]

Discrete-Time Fourier Transform

- Convergence Condition - An infinite series of the form
  \[ X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \]
  may or may not converge
- Let
  \[ X_K(e^{j\omega}) = \sum_{n=-K}^{K} x[n]e^{-j\omega n} \]

- Then for uniform convergence of \( X(e^{j\omega}) \),
  \[ \lim_{K \to \infty} \left| X(e^{j\omega}) - X_K(e^{j\omega}) \right| = 0 \]
- Now, if \( x[n] \) is an absolutely summable sequence, i.e., if
  \[ \sum_{n=-\infty}^{\infty} |x[n]| < \infty \]
Discrete-Time Fourier Transform

- Then
  \[ |X(e^{j\omega})| = \left| \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \right| \leq \sum_{n=-\infty}^{\infty} |x[n]| < \infty \]
  for all values of \( \omega \)
- Thus, the absolute summability of \( x[n] \) is a sufficient condition for the existence of the DTFT \( X(e^{j\omega}) \)

Example - The sequence \( x[n] = \alpha^n \mu[n] \) for \( |\alpha| < 1 \) is absolutely summable as

\[
\sum_{n=-\infty}^{\infty} |x[n]| = \sum_{n=0}^{\infty} |\alpha^n| = \frac{1}{1-|\alpha|} < \infty
\]

and its DTFT \( X(e^{j\omega}) \) therefore converges to \( 1/(1-\alpha e^{-j\omega}) \) uniformly

Discrete-Time Fourier Transform

- Since
  \[
  \sum_{n=-\infty}^{\infty} |x[n]|^2 \leq \left( \sum_{n=-\infty}^{\infty} |x[n]| \right)^2
  \]
  an absolutely summable sequence has always a finite energy
- However, a finite-energy sequence is not necessarily absolutely summable

Example - The sequence

\[
x[n] = \begin{cases} 1/n, & n \geq 1 \\ 0, & n \leq 0 \end{cases}
\]

has a finite energy equal to

\[
\mathcal{E}_x = \sum_{n=1}^{\infty} \left( \frac{1}{n} \right)^2 = \frac{\pi^2}{6}
\]

- But, \( x[n] \) is not absolutely summable

Discrete-Time Fourier Transform

- To represent a finite energy sequence \( x[n] \) that is not absolutely summable by a DTFT \( X(e^{j\omega}) \), it is necessary to consider a mean-square convergence of \( X(e^{j\omega}) \):

\[
\lim_{K \to \infty} \frac{1}{\pi} \int \left| X(e^{j\omega}) - X_K(e^{j\omega}) \right|^2 d\omega = 0
\]

where

\[
X_K(e^{j\omega}) = \sum_{n=-K}^{K} x[n] e^{-j\omega n}
\]

Here, the total energy of the error

\[
X(e^{j\omega}) - X_K(e^{j\omega})
\]

must approach zero at each value of \( \omega \) as \( K \) goes to \( \infty \)
- In such a case, the absolute value of the error \( |X(e^{j\omega}) - X_K(e^{j\omega})| \) may not go to zero as \( K \) goes to \( \infty \) and the DTFT is no longer bounded
**Example** - Consider the DTFT

\[ H_{LP}(e^{j\omega}) = \begin{cases} 1, & 0 \leq |\omega| \leq \omega_c \\ 0, & \omega_c < |\omega| \leq \pi \end{cases} \]

shown below

\[ H_{LP}(e^{j\omega}) \]

\[ -\omega_c \quad 0 \quad \omega_c \quad \omega \]

The inverse DTFT of \( H_{LP}(e^{j\omega}) \) is given by

\[ h_{LP}[n] = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega \]

\[ = \frac{1}{2\pi} \left( e^{j\omega_c n} - e^{-j\omega_c n} \right) \sin \omega_c n \frac{\sin \pi n}{\pi n}, \quad -\infty < n < \infty \]

• The energy of \( h_{LP}[n] \) is given by \( \omega_c / \pi \)

• \( h_{LP}[n] \) is a finite-energy sequence, but it is not absolutely summable

As a result

\[ \sum_{n=-\infty}^{\infty} h_{LP}[n] e^{-j\omega n} = \sum_{n=-K}^{K} \frac{\sin \omega_c n}{\pi n} e^{-j\omega n} \]

does not uniformly converge to \( H_{LP}(e^{j\omega}) \) for all values of \( \omega \), but converges to \( H_{LP}(e^{j\omega}) \) in the mean-square sense

As can be seen from these plots, independent of the value of \( K \) there are ripples in the plot of \( H_{LP,K}(e^{j\omega}) \) around both sides of the point \( \omega = \omega_c \)

• The number of ripples increases as \( K \) increases with the height of the largest ripple remaining the same for all values of \( K \)
**Discrete-Time Fourier Transform**

- As $K$ goes to infinity, the condition
  \[
  \lim_{K \to \infty} \int_{-\pi}^{\pi} \left| H_{LP}(e^{j\omega}) - H_{LP,K}(e^{j\omega}) \right|^2 d\omega = 0
  \]
  holds indicating the convergence of $H_{LP,K}(e^{j\omega})$ to $H_{LP}(e^{j\omega})$.

- The oscillatory behavior of $H_{LP,K}(e^{j\omega})$ approximating $H_{LP}(e^{j\omega})$ in the mean-square sense at a point of discontinuity is known as the **Gibbs phenomenon**.

**Discrete-Time Fourier Transform**

- The DTFT can also be defined for a certain class of sequences which are neither absolutely summable nor square summable.

- Examples of such sequences are the unit step sequence $\mu[n]$, the sinusoidal sequence $\cos(\omega_0 n + \phi)$ and the exponential sequence $A e^{\alpha n}$.

- For this type of sequences, a DTFT representation is possible using the Dirac delta function $\delta(\omega)$.

**Discrete-Time Fourier Transform**

- **Example** - Consider the complex exponential sequence
  
  \[ x[n] = e^{j\omega_0 n} \]

- Its DTFT is given by
  
  \[ X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi \delta(\omega - \omega_0 + 2\pi k) \]

  where $\delta(\omega)$ is an impulse function of $\omega$ and
  
  \[-\pi \leq \omega_0 \leq \pi \]

**Discrete-Time Fourier Transform**

- The function
  
  \[ X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi \delta(\omega - \omega_0 + 2\pi k) \]

  is a periodic function of $\omega$ with a period $2\pi$ and is called a periodic impulse train.

- To verify that $X(e^{j\omega})$ given above is indeed the DTFT of $x[n] = e^{j\omega_0 n}$, we compute the inverse DTFT of $X(e^{j\omega})$.

**Discrete-Time Fourier Transform**

- Thus
  
  \[ x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{k=-\infty}^{\infty} 2\pi \delta(\omega - \omega_0 + 2\pi k) e^{j\omega_0 n} d\omega \]

  \[ = \frac{1}{2\pi} \int_{-\pi}^{\pi} \delta(\omega - \omega_0) e^{j\omega_0 n} d\omega = e^{j\omega_0 n} \]

  where we have used the sampling property of the impulse function $\delta(\omega)$. 

**Discrete-Time Fourier Transform**

- A Dirac delta function $\delta(\omega)$ is a function of $\omega$ with infinite height, zero width, and unit area.

- It is the limiting form of a unit area pulse function $P_\Delta(\omega)$ as $\Delta$ goes to zero satisfying
  
  \[ \lim_{\Delta \to 0} \int_{-\pi}^{\pi} P_\Delta(\omega) d\omega = \int_{-\pi}^{\pi} \delta(\omega) d\omega = 1 \]

- The function $P_\Delta(\omega)$ as $\Delta$ goes to zero satisfying
Commonly Used DTFT Pairs

<table>
<thead>
<tr>
<th>Sequence $\delta[n]$</th>
<th>DTFT $1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi k)$</td>
<td>$1$</td>
</tr>
<tr>
<td>$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_t + 2\pi k)$</td>
<td>$e^{j\omega t}$</td>
</tr>
<tr>
<td>$\frac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi \delta(\omega + 2\pi k)$</td>
<td>$\mu(n)$, $</td>
</tr>
</tbody>
</table>

DTFT Properties

- There are a number of important properties of the DTFT that are useful in signal processing applications.
- These are listed here without proof.
- Their proofs are quite straightforward.
- We illustrate the applications of some of the DTFT properties.

### Table 3.2: General Properties of DTFT

<table>
<thead>
<tr>
<th>Type of Property</th>
<th>Sequence $x[n]$</th>
<th>Discrete-Time Fourier Transform $X(e^{j\omega})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linearity</td>
<td>$ax[n] + bx[n]$</td>
<td>$aX(e^{j\omega}) + bX(e^{j\omega})$</td>
</tr>
<tr>
<td>Time-shifting</td>
<td>$x[n-k]$</td>
<td>$e^{-j\omega k}X(e^{j\omega})$</td>
</tr>
<tr>
<td>Frequency-shifting</td>
<td>$x[n]\cdot e^{j\omega_0 n}$</td>
<td>$X(e^{j\omega + j\omega_0})$</td>
</tr>
<tr>
<td>Differentiation in frequency</td>
<td>$nx[n]$</td>
<td>$\frac{d}{d\omega}X(e^{j\omega})$</td>
</tr>
<tr>
<td>Convolution</td>
<td>$x[n]h[n]$</td>
<td>$X(e^{j\omega})H(e^{j\omega})$</td>
</tr>
<tr>
<td>Modulation</td>
<td>$x[n]e^{j\theta n}$</td>
<td>$\frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\theta})e^{j\theta \omega} d\theta$</td>
</tr>
<tr>
<td>Parseval’s relation</td>
<td>$\sum_{n=-\infty}^{\infty}</td>
<td>x[n]</td>
</tr>
</tbody>
</table>

### Table 3.3: DTFT Properties: Symmetry Relations

- $x[n]$ is a complex sequence.

<table>
<thead>
<tr>
<th>Sequence $x[n]$</th>
<th>Discrete-Time Fourier Transform $X(e^{j\omega})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x[-n]$</td>
<td>$X(e^{-j\omega})$</td>
</tr>
<tr>
<td>$x^*(n)$</td>
<td>$X^*(-e^{j\omega})$</td>
</tr>
<tr>
<td>$x(n)$</td>
<td>$X(e^{j\omega})$</td>
</tr>
<tr>
<td>$x(n)e^{j\omega_0 n}$</td>
<td>$X(e^{j(\omega - \omega_0)})$</td>
</tr>
<tr>
<td>$x(n)e^{-j\theta n}$</td>
<td>$X(e^{j(\omega + \theta)})$</td>
</tr>
</tbody>
</table>

### Table 3.4: DTFT Properties: Symmetry Relations

- $x[n]$ is a real sequence.

<table>
<thead>
<tr>
<th>Sequence $x[n]$</th>
<th>Discrete-Time Fourier Transform $X(e^{j\omega})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x[n]$</td>
<td>$X(e^{j\omega}) = X^*(e^{-j\omega})$</td>
</tr>
<tr>
<td>$x_n[n]$</td>
<td>$X_n(e^{j\omega}) = X(e^{j\omega})$</td>
</tr>
<tr>
<td>$x_o[n]$</td>
<td>$X_o(e^{j\omega}) = X(-e^{j\omega})$</td>
</tr>
<tr>
<td>Symmetry relations</td>
<td>$X(e^{j\omega}) = X^*(-e^{j\omega})$</td>
</tr>
<tr>
<td>$</td>
<td>X(e^{j\omega})</td>
</tr>
</tbody>
</table>

Note: $x_n[n]$ and $x_o[n]$ denote the even and odd parts of $x[n]$, respectively.

$x[n]$ is a complex sequence.

$x[n]$ is a real sequence.