DISCRETE-TIME ANALYTIC SIGNAL WITH IMPROVED SHIFTABILITY

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Abstract

In this thesis, we propose two methods for generating a discrete-time analytic (DTA) signal. The first is in the frequency domain and the second in the time domain.

We consider a common procedure in MATLAB for generating DTA signals and show how it fails for specific discrete-time real signals. A new frequency domain technique is formulated that resolves this defect. Both methods have the same redundancy. The new analytic signal preserves the original signal (real part) as also the zeros of its discrete spectrum in the negative frequencies. The superiority of the new method is in the introduction of one additional zero of the continuous spectrum of the original signal at a negative frequency. There is a corresponding improvement in shiftability but at a loss of orthogonality.

We also formulate a new time-domain method for DTA signal generation. The advantage of this technique is that is provides linear phase, orthogonality, invertibility and real-time implementation. Furthermore, we can also obtain improved shiftability. To our knowledge, no other technique for DTA generation provides all these properties.
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Notation and Acronyms

- \( j \) denote the complex number: \( j^2 = -1 \).

- The Fourier Transform of a continuous function will be denoted by its corresponding upper-case and the variable is denoted by \( \omega \). Example: \( F(\omega) \) is the Fourier transform of the continuous function \( f(t) \).

- The Discrete Time Fourier Transform (DTFT) of a sequence will be denoted by its corresponding upper-case and the variable is denoted by \( e^{j\omega} \). Example: \( X(e^{j\omega}) \) is the Discrete Time Fourier Transform of the sequence \( x(n) \).

- The Discrete Fourier Transform (DFT) of a sequence will be denoted by its corresponding upper-case and the variable is denoted by \( k \). Example: \( X(k) \) is the Discrete Fourier Transform of the sequence \( x(n) \).

- The Inverse Discrete Fourier Transform will be denoted by IDFT.

- By DTA we denote the discrete time analytic signal.

- By DTAF we denote the discrete time analytic filter introduced in chapter 3.
Chapter 1

Introduction

1.1 Background and motivation

Analytic signals, both continuous and discrete, are signals of considerable importance in many fields and especially in signal processing. Instead of processing a given real signal, it is often advantageous to consider its analytic counterpart - that which contains in its real part the original signal and in its imaginary part, a 90° phase shifted version (of the original signal). The imaginary part is referred to as the Hilbert transform of the real part and the filter generating the imaginary part is referred to as the Hilbert transformer or equivalently, a 90° phase shifter. The analytic signal is then a generalization of the complex exponential. Such signals are of use in a host of applications in signal processing and in communications. Examples include single-sideband analog communication systems and analog frequency-division multiplex system, to name just a few.

Discrete-time analytic (DTA) signals have characteristics similar to their analog counterparts. Their specific properties will be described later. They have found use in many applications such as: complex wavelets[5], radar[11], estimation of the
instantaneous frequency[2], feature detection[10],[16], texture segmentation[3], estimation of a signal in noise[17], damage diagnosis of machine[4], vibration analysis[12] etc. Hardware implementation also have been realized[8].

In this thesis we look at the generation of DTA signals and suggest better methods for the same. Methods currently in use are either frequency-based ones [13] or time-domain [19], [6] filtering methods. The former consists of setting the negative frequencies of the given signal to zero via the discrete Fourier transform (DFT), and subsequent generation of the DTA through an inverse DFT. The time-domain method consists of designing a half-band lowpass filter (one which passes frequencies between $-\pi/2, +\pi/2$) and a spectrum shift to the right by $\pi/2$ radians, maintaining high attenuation in the negative frequency band. This filter is referred to as a complex half-band filter. The output of such a filter to a given real signal generates the corresponding DTA signal. In the time domain approach, the length of the filter affects the accuracy of the approximation to the analytic signal. For instance in [19], the number of taps used is 128 which make this method inappropriate for small-length signals.

We formulate, in this thesis, two methods for the generation of DTA signals. We first show that the standard frequency domain method [13] fails to generate analytic signals for a certain class of signals. We expand on the frequency domain method to resolve that dilemma. However, in the process of that solution, we discover a more significant effect: by proper design, the new method can perform better for all classes of signals, as measured by a certain criteria known as “shiftability” [22].

The cost for such an improvement is a loss of orthogonality (between the real and imaginary parts of the DTA signal). The frequency domain approach is classified as “non-real-time” in that determination of the DTA signal of a given real input signal requires that the latter be fully available. This does not, of course, apply to a
corresponding design of DTA filters by this method, which can be designed off-line and used in real-time application.

The second design method for DTA signals is in the time-domain. This falls in the class of other time-domain methods for DTA signal generation in that a DTA filter is first designed. A DTA signal of a given real signal is then obtained as the output to such a filter. The advantage of this method is that the DTA signal has the proprieties of (i) linear phase (ii) orthogonality (iii) preservation of the original real valued signal and (iv) reduction in shiftability. Methods currently used do not satisfy at least one of these properties.

In this chapter, we present a brief introduction to analytic signals as also reference to material contained in this thesis.

1.2 Hilbert transform pair

Consider a real valued signal \( f(t) \). The Hilbert transform pair \([7]\) is defined by the integrals

\[
\hat{H}\{f(t)\} = \frac{-1}{\pi} P \int_{-\infty}^{+\infty} \frac{f(\eta)}{\eta - t} d\eta \quad (1.1)
\]

\[
f(t) = \frac{1}{\pi} P \int_{-\infty}^{+\infty} \frac{\hat{H}\{f(t)\}}{\eta - t} d\eta. \quad (1.2)
\]

The integrals in these equations are improper integrals in the sense of the Cauchy principal value denoted by \( P \). Equation (1.1), for instance, is defined by the limit

\[
\hat{H}\{f(t)\} = \lim_{\epsilon \to 0; A \to \infty} \left\{ \frac{-1}{\pi} \int_{-A}^{t-\epsilon} \frac{f(\eta)}{\eta - t} d\eta + \frac{-1}{\pi} \int_{t+\epsilon}^{A} \frac{f(\eta)}{\eta - t} d\eta \right\}.
\]
The Hilbert transform pair (1.1) and (1.2) are often written in terms of the convolution notation

\[ \hat{H}\{f(t)\} = \frac{1}{\pi t} * f(t) \]  
(1.3)

\[ f(t) = \frac{-1}{\pi t} * H\{f(t)\}. \]  
(1.4)

\(f(t)\) and \(\hat{H}\{f(t)\}\) given by equations (1.3) and (1.4) are also called a Hilbert transform pair.

1.3 Properties of the Hilbert transform

- **Linearity**

The Hilbert Transform is a linear transform

\[ \hat{H}\{c_1 f_1(t) + c_2 f_2(t)\} = c_1 \hat{H}\{f_1(t)\} + c_2 \hat{H}\{f_2(t)\} \]

where \(f_1(t)\) and \(f_2(t)\) are real valued function and \(c_1\) and \(c_2\) are real constants.

- **Differentiability**

\[ \hat{H}\left\{ \frac{d}{dt} f(t) \right\} = \frac{d}{dt} \{\hat{H}\{f(t)\}\} \]

Where \(f(t)\) is a real valued function whose derivative exists.

- **Multiple Hilbert Transform**

\[ \hat{H}\{\hat{H}\{f(t)\}\} = -f(t) \]

where \(f(t)\) is a real valued function.
• Orthogonality

A real function and its Hilbert Transform are orthogonal.

\[ \int_{-\infty}^{+\infty} f(t) \hat{H}\{f(t)\} dt = 0 \]

1.4 Analytic functions

The complex signal whose imaginary part is the Hilbert transform of the real part is called an analytic signal [7]. Consider a real signal \( f(t) \). The complex signal \( h(t) \) defined by equation (1.5) is an analytic signal.

\[ h(t) = f(t) + j \hat{H}\{f(t)\} \quad (1.5) \]

where \( j^2 = -1 \).

The term analytic is used in the sense of a complex function \( \psi(z) \) of the complex variable \( z = t + j\tau \). If we take \( t \) and \( \tau \) as the rectangular coordinates in the complex plane \( C \) and take a domain \( D \) in this plane. Define a rule between each point in \( D \) and a complex number \( \omega \), and a complex function \( \psi(z) \). This function can be written as

\[ \psi(z) = \psi(t, \tau) = f(t, \tau) + j g(t, \tau) \]

where \( f \) and \( g \) are two real valued functions in the domain \( D \). The function \( \psi(z) \) is analytic in the domain \( D \) if and only if \( f \) and \( g \) satisfy the Cauchy-Riemann equations

\[ \frac{\partial f(t, \tau)}{\partial t} = \frac{\partial g(t, \tau)}{\partial \tau} \]

\[ \frac{\partial f(t, \tau)}{\partial \tau} = -\frac{\partial g(t, \tau)}{\partial t} . \]
1.5 The Fourier transform and analytic functions

The Fourier transform pair [18] of a real-valued signal $f(t)$ is defined by

$$F(\omega) = \int_{-\infty}^{+\infty} f(t) e^{-j\omega t} dt$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\omega) e^{j\omega t} d\omega$$

Now consider the analytic function $z(t) = f(t) + j\hat{H}\{f(t)\}$. It is easily seen that the Fourier transform $Z(\omega)$ of $z(t)$.

$$Z(\omega) = \begin{cases} 
2F(\omega), & \omega > 0 \\
F(0), & \omega = 0 \\
0, & \omega < 0.
\end{cases}$$

Therefore an analytic signal $z(t)$ is a complex signal where $Z(\omega) = 0$ for $\omega < 0$.

1.6 Discrete-time analytic signals

As described earlier, a continuous-time analytic signal is a complex time function having a Fourier transform equal to zero for all $\omega < 0$. For a discrete-time sequence, we cannot require the same constraint since the discrete-time Fourier transform (DTFT) spectrum is periodic. Instead, a discrete-time sequence is defined to be "analytic" [18] by requiring that its discrete-time Fourier transform (DTFT) vanish in the interval $[-\pi, 0)$. Such a sequence will henceforth be denoted as a discrete-time analytic (DTA) signal.

The concept of a DTA signal was formulated in [18]. Let $h(n)$ be the impulse
response of a linear time invariant system such that its DTFT $H(e^{j\omega})$ is given by

$$H(e^{j\omega}) = \begin{cases} -j, & 0 < \omega < \pi \\ j, & -\pi \leq \omega < 0 \end{cases} \quad (1.7)$$

A system with a DTFT given by equation (1.7) is called a Hilbert transformer or equivalently, an ideal 90-degree phase shifter. Denote by $\bar{H}\{x(n)\}$ the output of the system given by equation (1.7) to the input $x(n)$. Therefore, for $x(n)$ a finite real valued sequence, the DTA signal $z(n)$ is defined as

$$z(n) = x(n) + j\bar{H}\{x(n)\}.$$  

The impulse response $h(n)$ of a 90-degree phase shifter corresponding to the frequency response given by equation (1.7), is

$$h(n) = \begin{cases} \frac{2 \sin^2(\pi n/2)}{\pi n}, & n \neq 0 \\ 0, & n = 0 \end{cases} \quad (1.8)$$

The impulse response of the Hilbert transformer given by (1.8) is a two-sided infinite-length impulse response. Consequently causal implementation requires approximation. Two methods are described in [18, Sec.10.4]. Finite impulse response (FIR) approximation with constant group delay can be obtained using either termination of the Fourier series using windows or minimization using some frequency-weighted criteria. The first one consists of using a Kaiser window, while the second uses an equiripple approximation. In both cases an additional linear phase shift is generated for causal FIR filter implementation.
1.7 Overview of this work

This thesis considers the problem of generation of DTA signals from discrete-time real valued signals. We introduce two new methods. The first method - a frequency domain method - is introduced in Chapter 2. The second method, which is a time domain approach, is described in Chapter 3.

In Chapter 2, we consider the algorithm used in [13] and show that it fails to generate a DTA for a certain class of discrete-time real signal. We present a new method based on the DTFT, which preserves the original signal (real part) and show that it also preserves the zeros of its discrete spectrum at the negative frequencies. The superiority of the new method is in the introduction of one additional zero of the continuous spectrum of the original signal at a negative frequency. This resolves the problem encountered in [13]. More significantly, new features are discovered: application to all classes of signals results in an improvement in performance, as measured by the concept of shiftability [22], which we describe later. Both methods are compared empirically, using the same concept of shiftability. The cost of the new method is a loss of orthogonality.

Currently used time-domain methods for generation of DTA signals [19], [5], [18] are reviewed in Chapter 3. We show that they lack at least one of the following important properties: generalized linear phase, preservation of the original signal, orthogonality and real-time implementation. A new method, based on a very simple idea, is proposed in Chapter 3. We show that it satisfies all the proprieties stated above. We make empirical comparison of the new method to that of [5] using shiftability and show gains in shiftability. Sacrificing orthogonality by applying our method in Chapter 2, provides yet further gains.

We conclude in Chapter 4 and provide a very brief overview of applications where our techniques for DTA signal generation could be utilized.
Chapter 2

New Method: Frequency domain approach

2.1 Introduction

We consider a common frequency domain procedure for generating discrete-time analytic signals and show how it fails for specific signals. A new frequency domain technique is formulated that solves the defect. Both methods have the same redundancy. The new analytic signal preserves the original signal (real part) as also the zeros of its discrete spectrum in the negative frequencies. The advantage of the method is in the introduction of one additional zero of the continuous spectrum of the analytic signal at a negative frequency and a corresponding improvement in shiftability.
2.2 Frequency domain approach

For \( x(n) \) a finite real valued sequence, the DTA signal \( z(n) \) is defined as

\[
z(n) = x(n) + jH\{x(n)\},
\]

where \( H \) is the Hilbert transform operator, and \( j^2 = -1 \). The periodic spectrum of \( z(n) \) is

\[
Z(e^{j\omega}) = \sum_{n=0}^{N-1} z(n)e^{-j\omega n}.
\]

where \( N \) is the length of \( z(n) \). The DTFT is periodic with period \( 2\pi \). Thus \( \omega \) is considered in the interval \([-\pi, \pi]\). For analyticity it is required ([18], Sec.10.4) that \( Z(e^{j\omega}) = 0 \) for \( \omega \in [-\pi, 0) \). Because \( \omega \) is continuous in the interval \([-\pi, \pi]\), the DTFT cannot be computed exactly. Hence the necessity for using the DFT. The DFT is obtained by uniformly sampling the DTFT on the \( \omega \)-axis at \( \omega_k = 2\pi k/N \), where \( 0 \leq k \leq N - 1 \). Thus

\[
Z(k) = Z(e^{j\omega})|_{\omega = \frac{2\pi k}{N}} = \sum_{n=0}^{N-1} z(n)e^{-j2\pi kn/N}.
\]

A common approach to generating a DTA signal resides in the frequency domain [13], [7]. We assume \( N \) to be even (the case for \( N \) odd is easily handled (see [7], page 145)). The procedure for deriving the discrete analytic signal consists of three steps:

- Compute the \( N \)-point DFT of \( x(n) \).
- Form the \( N \)-point DFT of the corresponding DTA signal by multiplying the \( N \)-point DFT of \( x(n) \) by the vector:
\[ a(n) = \begin{cases} 
1, & n = 0 \\
2, & 1 \leq n \leq N/2 - 1, \\
1, & n = N/2 \\
0, & N/2 + 1 \leq n \leq N - 1. 
\end{cases} \]

Thus, the \( N \)-point DFT of the discrete analytic signal is

\[
Z(k) = \begin{cases} 
X(k), & k = 0 \\
2X(k), & 1 \leq k \leq N/2 - 1 \\
X(k), & k = N/2 \\
0, & N/2 + 1 \leq k \leq N - 1. 
\end{cases} \tag{2.1}
\]

- Obtain the DTA signal by computing the inverse DFT of the \( N \)-point DFT:

\[
z(n) = 1/N \sum_{k=0}^{N-1} Z(k)e^{j2\pi kn/N}. \tag{2.2}
\]

Based on this procedure, we now show that the corresponding DTA signal can be written as

\[
z(n) = x(n) + j(2/N) \sum_{p=0}^{N/2-1} x(2p + 1) \cot(\pi(n - (2p + 1))/N), \quad \text{if } n \text{ odd} \tag{2.3}
\]

\[
z(n) = x(n) + j(2/N) \sum_{p=0}^{N/2-1} x(2p) \cot(\pi(n - 2p)/N), \quad \text{if } n \text{ even}. \tag{2.4}
\]

Our proof proceeds as follows: From equation (2.2) and using equation (2.1) we have
\[ z(n) = \frac{1}{N} \sum_{k=0}^{N/2} Z(k)e^{j2\pi kn/N} \]

\[ = \frac{1}{N} Z(0) + \frac{1}{N} \sum_{k=1}^{N/2-1} Z(k)e^{j2\pi kn/N} + \frac{1}{N} Z(N/2)e^{j2\pi(2N)n/N} \]

\[ = \frac{1}{N} X(0) + 2/N \sum_{k=1}^{N/2-1} X(k)e^{j2\pi kn/N} + \frac{1}{N} X(N/2)e^{j\pi n} \]

\[ = \frac{1}{N} X(0) + 2/N \sum_{k=1}^{N/2-1} X(k)e^{j2\pi kn/N} + \frac{1}{N} X(N/2)e^{j\pi n} + x(n) - x(n) \]

\[ = x(n) + \frac{1}{N} X(0) + 2/N \sum_{k=1}^{N/2-1} X(k)e^{j2\pi kn/N} + \frac{1}{N} X(N/2)e^{j\pi n} \]

\[-1/N \sum_{k=0}^{N-1} X(k) e^{j2\pi kn/N} \]

\[ = x(n) + \frac{1}{N} X(0) + 2/N \sum_{k=1}^{N/2-1} X(k)e^{j2\pi kn/N} - 1/N \sum_{k=0}^{N/2-1} X(k)e^{j2\pi kn/N} \]

\[-1/N \sum_{k=N/2+1}^{N-1} X(k)e^{j2\pi kn/N} \]

\[ = x(n) + 1/N \sum_{k=1}^{N/2-1} X(k)e^{j2\pi kn/N} - 1/N \sum_{k=N/2+1}^{N-1} X(k)e^{j2\pi kn/N}. \tag{2.5} \]

Let \( B \equiv -1/N \sum_{k=N/2+1}^{N-1} X(k)e^{j2\pi kn/N} \) and change variable \( k = p + \frac{N}{2} \). Thus

\[ B = -1/N \sum_{p=1}^{N/2-1} X(p + \frac{N}{2})e^{j2\pi (p+\frac{N}{2})m/N} = -1/N \sum_{p=1}^{N/2-1} X(p + \frac{N}{2})e^{j2\pi pm/N}e^{j\pi n} \]

\[ = -1/N \sum_{p=1}^{N/2-1} X(p + \frac{N}{2})e^{j2\pi pm/N}(-1)^n. \]

We therefore have from equation (2.5)

\[ z(n) = x(n) + 1/N \sum_{k=1}^{N/2-1} X(k)e^{j2\pi kn/N} - 1/N \sum_{k=1}^{N/2-1} X(k + \frac{N}{2})e^{j2\pi kn/N}(-1)^n \]
\[ = x(n) + 1/N \sum_{k=1}^{N/2-1} \{ X(k)e^{j2\pi kN/N} - X(k + N/2)e^{j2\pi kN/N}(-1)^n \} \]
\[ = x(n) + 1/N \sum_{k=1}^{N/2-1} e^{j2\pi kN/N} \{ X(k) - X(k + N/2)(-1)^n \}. \quad (2.6) \]

The terms in parenthesis in equation (2.6) is
\[ X(k) - X(k + \frac{N}{2})(-1)^n = \sum_{m=0}^{N-1} x(m)e^{-j2\pi mk/N} - \sum_{m=0}^{N-1} x(m)e^{-j2\pi (k+N/2)m/N}(-1)^m \]
\[ = \sum_{m=0}^{N-1} x(m)\{ e^{-j2\pi mk/N} - e^{-j2\pi (k+N/2)m/N}(-1)^m \} \]
\[ = \sum_{m=0}^{N-1} x(m)e^{-j2\pi mk/N}\{ 1 - (-1)^{m+n} \}. \]

Accordingly, equations (2.6) is written as
\[ z(n) = x(n) + 1/N \sum_{k=1}^{N/2-1} e^{j2\pi kN/N} \{ \sum_{m=0}^{N-1} x(m)e^{-j2\pi mk/N}\{ 1 - (-1)^{m+n} \} \} \]
\[ = x(n) + 1/N\{ \sum_{m=0}^{N-1} x(m)(1 - (-1)^{m+n}) \sum_{k=1}^{N/2-1} e^{j2\pi k(n-m)/N} \}. \quad (2.7) \]

Observe that in equation (2.7) if \((m + n)\) is even the term
\[ x(m)(1 - (-1)^{m+n}) \sum_{k=1}^{N/2-1} e^{j2\pi k(n-m)/N} = 0; \]

therefore we are interested in the case when \(m + n\) is odd. Suppose \((m + n)\) is odd. We first simplify the expression \(\sum_{k=1}^{N/2-1} e^{j2\pi k(n-m)/N}\) in equation (2.7) as
\[ \sum_{k=1}^{N/2-1} e^{j2\pi k(n-m)/N} = \sum_{k=0}^{N/2-1} e^{j2\pi k(n-m)/N} - 1 = \frac{1 - e^{j\pi(n-m)}}{1 - e^{j2\pi(n-m)/N}} - 1 \]
\[ = \frac{1 - (-1)^{(n-m)}}{1 - e^{j2\pi(n-m)/N}} - 1. \]
Now \((m + n)\) odd is equivalent to \((n - m)\) odd. Therefore

\[
\sum_{k=1}^{N/2-1} e^{j2\pi k(n-m)/N} = \frac{2}{1 - e^{j2\pi (n-m)/N}} - 1 = \frac{1 + e^{j2\pi (n-m)/N}}{1 - e^{j2\pi (n-m)/N}}
\]

\[
= \frac{e^{-j\pi (n-m)/N} + e^{j\pi (n-m)/N}}{e^{-j\pi (n-m)/N} - e^{j\pi (n-m)/N}} = \frac{2 \cos(\frac{\pi}{N}(n-m))}{-j2\sin(\frac{\pi}{N}(n-m))}
\]

\[
= j\frac{\cos(\frac{\pi}{N}(n-m))}{\sin(\frac{\pi}{N}(n-m))} = j\cot(\frac{\pi}{N}(n-m)), \tag{2.8}
\]

For \(n\) even, \((n - m)\) odd is equivalent to \(m\) odd. Also for \(n\) odd, \((n - m)\) odd is equivalent to \(m\) even. Therefore, for \(n\) even and \(m\) odd, we have \(1 - (-1)^{n+m} = 2\), and for \(n\) even and \(m\) even, we have \(1 - (-1)^{n+m} = 0\). Thus, referring back to equation (2.7) and using (2.8), we conclude that for \(n\) even

\[
z(n) = x(n) + 1/N \sum_{m=0}^{N-1} x(m)(1 - (-1)^{m+n})j\cot(\frac{\pi}{N}(n-m))
\]

\[
= x(n) + 1/N \sum_{m=0, m \text{ odd}}^{N-1} x(m)(1 - (-1)^{m+n})j\cot(\frac{\pi}{N}(n-m))
\]

\[
= x(n) + j(2/N) \sum_{p=0}^{N/2-1} x(2p+1) \cot(\pi(n - (2p+1))/N).
\]

Similarly for \(n\) odd, we can show that

\[
z(n) = x(n) + j(2/N) \sum_{p=0}^{N/2-1} x(2p) \cot(\pi(n - 2p)/N).
\]

This concludes the proof.

2.2.1 Problem with the current method

As stated earlier, a continuous-time analytic signal is defined as that with spectrum that is zero for \(\omega < 0\). Hence it is a complex signal. Thus a non-zero real signal is not analytic. Similarly, a non-zero discrete-time real signal is not analytic. From
equations (2.3) and (2.4), we observe that were the imaginary part of $z(n)$ to be zero, then $z(n)$ would be real and consequently not analytic. Hence consider the following situation: Suppose all even values of $x(n)$ are equal to some constant $\alpha$ and all odd values to some constant $\beta$ (with no loss of generality we assume both constants to be different from zero). Then we have from equations (2.3) and (2.4)

$$\text{imag}(z(n)) = \frac{2}{N} \beta \sum_{p=0}^{N/2-1} \cot(\pi(n - (2p+1))/N), \quad \text{if } n \text{ even}$$

$$\text{imag}(z(n)) = \frac{2}{N} \alpha \sum_{p=0}^{N/2-1} \cot(\pi(n - 2p)/N), \quad \text{if } n \text{ odd.}$$

For the imaginary part of $z(n)$ to be zero, we must have

$$\sum_{p=0}^{N/2-1} \cot(\pi(n - (2p+1))/N) = 0, \quad \text{if } n \text{ even} \quad (2.9)$$

and

$$\sum_{p=0}^{N/2-1} \cot(\pi(n - 2p)/N) = 0, \quad \text{if } n \text{ odd.} \quad (2.10)$$

We show that the conditions in equations (2.9), (2.10) are always true. The proof is in two steps. It is first shown that each equation can be written as a circulant matrix; then, the first row of each matrix sums to zero. Equation (2.9) is proven first. Denote by $A = (a)_{2p}$ the matrix whose elements are $(a)_{2p} = \cot(\pi(n - (2p+1))/N)$, where
\[ 0 \leq p \leq \frac{N}{2} - 1, \ 0 \leq n \leq N - 1, \text{ and } n \text{ even. Hence} \]

\[
A = \begin{bmatrix}
  a_{00} & a_{01} & a_{02} & \cdots & \cdots & a_{0(N-2)(\frac{N}{2}-1)} \\
  a_{20} & a_{21} & a_{22} & a_{23} & \cdots & a_{2(N-2)(\frac{N}{2}-1)} \\
  \vdots & a_{41} & a_{42} & a_{43} & a_{44} & \cdots \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\
  a_{(N-2)0} & a_{(N-2)1} & \cdots & \cdots & a_{(N-2)(\frac{N}{2}-2)} & a_{(N-2)(\frac{N}{2}-1)}
\end{bmatrix} \quad (2.11)
\]

We need to show that the matrix \( A = (a)_{\frac{n}{2}p} \), for \( 0 \leq p \leq \frac{N}{2} - 1 \), \( 0 \leq n \leq N - 1 \) and \( n \) even is a circulant matrix. This is equivalent to showing that it is Toeplitz and the first column is obtained by transposing the first row and flipping the last \( (\frac{N}{2} - 1) \) terms. We show that the diagonal \( \frac{n}{2} \) is constant for \( n \) even and \( 0 \leq n \leq N - 1 \). Thus we need to show that \( a_{20} = a_{(\frac{n}{2}+p)p} \) where, \( 0 \leq p \leq \frac{N}{2} - 1 \), \( (\frac{n}{2} + p) \leq N - 1 \), \( 0 \leq n \leq N - 1 \), and \( n \) even. We have

\[
a_{(\frac{n}{2}+p)p} = \cot \left( \frac{\pi}{N} \left( 2 \left( \frac{n}{2} + p \right) - (2p + 1) \right) \right) \\
= \cot \left( \frac{\pi}{N} (n - 1) \right) \\
= a_{20}.
\]

Now we show that the diagonal \( \frac{n}{2} \) is constant for \( n \) even and \( 0 \leq n \leq N - 1 \) which is equivalent to showing that \( a_{20} = a_{(\frac{n}{2}+p)p} \) where, \( 0 \leq p \leq \frac{N}{2} - 1 \), \( (\frac{n}{2} + p) \leq N - 1 \), \( 0 \leq n \leq N - 1 \),
\[ 0 \leq n \leq N - 1, \text{ and } n \text{ even.} \]

\[
a_p(p + \frac{N}{2}) = \cot\left(\frac{\pi}{N}(2p - (2p + 2\frac{n}{2} + 1))\right) \\
= \cot\left(\frac{\pi}{N}(-(n - 1))\right) \\
= a_{0\frac{N}{2}}.
\]

Therefore the main diagonal and each subdiagonal of \( A \) is constant. Next, we need to verify that given the first row, the first column is obtained by transposing the first row and flipping the last \( \frac{N}{2} - 1 \) terms. This is equivalent to verifying that \( a_{p0} = a_{0\frac{N}{2} - p} \) where \( 0 \leq p \leq \frac{N}{2} - 1 \). We see that

\[
a_{0\frac{N}{2} - p} = \cot\left(\frac{\pi}{N}(-2\left(\frac{N}{2} - p\right) + 1)\right) \\
= \cot\left(\frac{\pi}{N}(2p - 1 - \pi)\right) \\
= \cot\left(\frac{\pi}{N}(2p - 1)\right) \\
= a_{p0}.
\]

Therefore we can conclude that \( A \) is circulant. Now, in order to show

\[
\sum_{p=0}^{N/2-1} \cot\left(\pi(n - (2p + 1))/N\right) = 0, \quad \forall \ 0 \leq n \leq (N - 1)
\]

and \( n \) even, it is enough to show that the first row of the matrix \( A \) sums to zero.

For \( n = 0 \) we get

\[
\sum_{p=0}^{N/2-1} \cot\left(\pi(-(2p + 1))/N\right) = \sum_{p=0}^{N/4-1} \cot\left(\pi(-(2p + 1))/N\right) + \sum_{p=N/4}^{N/2-1} \cot\left(\pi(-(2p + 1))/N\right).
\]

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Now
\[ \sum_{p=N/4}^{N/2-1} \cot(\pi(-(2p + 1))/N) = \sum_{q=0}^{N/4-1} \cot(\frac{\pi}{N}(-2(\frac{N}{2} - 1 - q) + 1)) \]
\[ = \sum_{q=0}^{N/4-1} \cot(\frac{\pi}{N}(2q + 1) - \pi) = \sum_{q=0}^{N/4-1} \cot(\frac{\pi}{N}(2q + 1)). \]

Therefore
\[ \sum_{p=0}^{N/2-1} \cot(\pi(-(2p + 1))/N) = \sum_{p=0}^{N/4-1} \cot(\pi(-(2p + 1))/N) + \sum_{q=0}^{N/4-1} \cot(\frac{\pi}{N}(2q + 1)) \]
\[ = 0. \]

Hence, \( \forall \ 0 \leq n \leq (N-1) \) and \( n \) even we have \( \sum_{p=0}^{N/2-1} \cot(\pi(-(2p + 1))/N) = 0. \)
Likewise, we can show that \( \sum_{p=0}^{N/2-1} \cot(\pi(n - 2p)/N) = 0, \ \forall \ 0 \leq n \leq (N-1) \) and \( n \) odd. Accordingly, we conclude that for this specific class of signals, \( z(n) \) is real and hence the method fails to generate a DTA signal.

**Example:** Let \( N = 4 \). Let \( x(n) = [x(0) \ x(1) \ x(2) \ x(3)] \) be a discrete real valued signal. Thus from equations (2.3), (2.4) the DTA \( z(n) \) corresponding to \( x(n) \) is
\[ z(0) = x(0) + \frac{j}{2}(x(3) - x(1)) \]
\[ z(1) = x(1) + \frac{j}{2}(x(0) - x(2)) \]
\[ z(2) = x(2) + \frac{j}{2}(x(1) - x(3)) \]
\[ z(3) = x(3) + \frac{j}{2}(x(2) - x(0)). \]

For \( x(3) = x(1) = 1 \) and \( x(2) = x(0) = 2 \) we get \( z(n) = x(n). \) Also, using MATLAB6.5, which uses the same algorithm [13] we see that:
\[
\text{hilbert}([1 \ 2 \ 1 \ 2]) = [1 \ 2 \ 1 \ 2];
\]
Note that in MATLAB6.5, the DTA signal is generated by the function \textit{hilbert}.

2.3 The new method

We have seen an example where the algorithm in [13] fails to generate the corresponding DTA signal. The new method for resolving this dilemma builds on the DTA signal formulation of equations (2.3) and (2.4). We make a simple modification to the imaginary part while ensuring that the real part is unchanged. This, as we will observe, becomes a frequency domain approach for which the algorithm in [13] is a special case. We label the new method \textit{ehilbert}. As we will observe, the procedure in the frequency domain results in the addition of an imaginary number to the DC and the Nyquist terms of the DTA signal obtained using \textit{hilbert}. This guarantees that the DTFT (and hence the DFT) equals zero at $\omega_k = 2\pi k/N$, $N/2 + 1 \leq k \leq N - 1$. In addition, the value of the DTFT can be made zero at \textit{one other} negative frequency of our choice. This resolves the dilemma faced earlier and also forces the continuous spectrum to be small around a region of a point in $[-\pi, 0)$, thereby aiding its shiftability property. (The latter property has yet to be defined.) In addition, we have ensured that the real part of the corresponding analytic signal is the original signal. We proceed as follows:

Let $x(n)$ be a finite real valued sequence of length $N$. We restrict our attention to only the case where $N$ is even. With reference to equations (2.3) and (2.4) let $s(n)$ be an analytic signal such that for $n$ even

$$s(n) = x(n) + j(2/N)\{\sum_{p=0}^{N/2-1} x(2p+1)\cot(\pi(n-(2p+1))/N) + a\}. \quad (2.12)$$
and for $n$ odd

$$s(n) = x(n) + j(2/N)\left\{ \sum_{p=0}^{N/2-1} x(2p) \cot(\pi(n - 2p)/N) + b \right\},$$

(2.13)

where we have added real variables $a$ and $b$ to the imaginary parts of the analytic signal $Z(N)$. We will show that the DFT, $S(k)$, of $s(n)$ is indeed zero for negative frequencies defined by $N/2 + 1 \leq k \leq N - 1$. For $n$ even

$$s(n) = z(n) + j(2/N)a = z(n) + jt(n)$$

and for $n$ odd

$$s(n) = z(n) + j(2/N)b = z(n) + jt(n)$$

where

$$t(n) = \begin{cases} 2a/N, & n \text{ even} \\ 2b/N, & n \text{ odd}. \end{cases}$$

The DFT of $t(n)$ is equal to

$$T(k) = \sum_{n=0}^{N-1} t(n)e^{-j2\pi kn/N}$$

$$= \sum_{n=0}^{N/2-1} t(2n)e^{-j2\pi k2n/N} + \sum_{n=0}^{N/2-1} t(2n + 1)e^{-j2\pi k(2n+1)/N}$$

$$= \frac{2}{N} \sum_{n=0}^{N/2-1} e^{-j2\pi k2n/N} + b \sum_{n=0}^{N/2-1} e^{-j2\pi k(2n+1)/N}.$$ 

Thus

$$S(k) = Z(k) + jT(k)$$
\[ Z(K) + j\{2/N\left\{a \sum_{n=0}^{N/2-1} e^{-j2\pi k 2n/N} + b \sum_{n=0}^{N/2-1} e^{-j2\pi k(2n+1)/N}\right\}\} \]

where \(Z(k)\) is the DFT of \(z(n)\). We now show that

\[ S(k) = \begin{cases} 
X(k) + j(a+b), & k = 0 \\
2X(k), & 1 \leq k \leq N/2 - 1 \\
X(k) + j(a-b), & k = N/2 \\
0, & N/2 + 1 \leq k \leq N - 1.
\] (2.14)

Recall that \(Z(k)\) satisfies the following

\[ Z(k) = \begin{cases} 
X(k), & k = 0 \\
2X(k), & 1 \leq k \leq N/2 - 1 \\
X(k), & k = N/2 \\
0, & N/2 + 1 \leq k \leq N - 1.
\]

Therefore it is sufficient to show that

\[ T(k) = \begin{cases} 
a + b, & k = 0 \\
0, & 1 \leq k \leq N/2 - 1 \\
a - b, & k = N/2 \\
0, & N/2 + 1 \leq k \leq N - 1.
\] (2.15)

The proofs for \(k = 0\) and \(k = N/2\) are straightforward. For \(1 \leq k \leq N/2 - 1\) it is easily verified that

\[ \sum_{n=0}^{N/2-1} e^{-j2\pi k 2n/N} = \sum_{n=0}^{N/2-1} e^{-j2\pi k(2n+1)/N} = 0. \]

For the remaining interval the proof follows by the conjugate symmetry relationship for real signals. Having established equation (2.14), where \(S(k)\) has a one-sided
spectrum like $Z(k)$, parameters $a$ and $b$ are available to establish other effects. We utilize this degree of freedom, that is, we choose values for $a$ and $b$ to force the DTFT of $s(n)$ to be zero for some $\omega$ in the interval $(-\pi, 0)$. This will alleviate the problem encountered before, by generating a complex signal rather than a real one for the special class of signals referred to earlier. Zeroing the DTFT for some $\omega$ in the negative frequency range will also form a neighborhood of $\omega$ where the DTFT could be very small, thus allowing for improved shiftability. Hence, for some $\omega$, we need to determine $a$ and $b$ such that

\[ S(e^{j\omega}) = \sum_{n=0}^{N-1} s(n)e^{-j\omega n} = 0 \]  \hspace{1cm} (2.16)

where $s(n)$ is defined by equations (2.12) and (2.13). We first need to establish that a solution exists. To do that, we proceed as follows: we equate the real and imaginary part of equation (2.16) to zero. We make the following identification for the real parts:

\[ r_1(\omega) = 2/N \sum_{p=0}^{N/2-1} x(2p) \cos(2p\omega) \]

\[ \alpha_1(\omega) = \sum_{p=0}^{N/2-1} x(2p) \cos(2p\omega) \]

\[ \alpha_2(\omega) = 2/N \sum_{p=0}^{N/2-1} \sum_{q=0}^{N/2-1} x(2q+1) \cot(f(p,q)) \sin(2p\omega) \]

\[ r_2(\omega) = 2/N \sum_{p=1}^{N/2} \sin(\omega(2p - 1)) \]

\[ \alpha_3(\omega) = \sum_{p=1}^{N/2} x(2p - 1) \cos(\omega(2p - 1)) \]

\[ \alpha_4(\omega) = 2/N \sum_{p=1}^{N/2} \sum_{q=0}^{N/2-1} x(2q) \cot(f(p,q)) \sin(\omega(2p - 1)). \]

and the following for the imaginary parts:

\[ r_{i1}(\omega) = 2/N \sum_{p=0}^{N/2-1} \cos(2p\omega) \]

\[ \alpha_{i1}(\omega) = -\sum_{p=0}^{N/2-1} x(2p) \sin(2p\omega) \]

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\[ \alpha_{22}(\omega) = -\frac{2}{N} \sum_{p=0}^{N/2-1} \sum_{q=0}^{N/2-1} x(2q + 1) \cot(f(p, q)) \cos(2p\omega) \]

\[ r_{i2}(\omega) = \frac{2}{N} \sum_{p=1}^{N/2} \cos(\omega(2p - 1)) \]

\[ \alpha_{i3}(\omega) = -\sum_{p=1}^{N/2} x(2p - 1) \sin(\omega(2p - 1)) \]

\[ \alpha_{i4}(\omega) = -\frac{2}{N} \sum_{p=1}^{N/2} \sum_{q=0}^{N/2-1} x(2q) \cot(f(p, q)) \cos(\omega(2p - 1)). \]

where \( f(p, q) = (\pi/N)(2p - (2q + 1)) \).

Hence equation (2.16) implies that

\[
\begin{align*}
\begin{cases}
\alpha_1(\omega) - b_2(\omega) &= \alpha_3(\omega) + \alpha_4(\omega) - \alpha_1(\omega) - \alpha_2(\omega) \\
\alpha_{i1}(\omega) - b_{i2}(\omega) &= \alpha_{i3}(\omega) + \alpha_{i4}(\omega) - \alpha_{i1}(\omega) - \alpha_{i2}(\omega)
\end{cases}
\end{align*}
\]

Accordingly, for \( \Delta(\omega) = r_{i2}(\omega)r_{i1}(\omega) - r_{i1}(\omega)r_{i2}(\omega) \neq 0 \), we have values for \( a \) and \( b \) as follows:

\[
\begin{align*}
a &= \frac{\{ -r_{i2}(\omega)(\alpha_3(\omega) + \alpha_4(\omega) - \alpha_1(\omega) - \alpha_2(\omega)) + r_2(\omega)(\alpha_{i3}(\omega) + \alpha_{i4}(\omega) - \alpha_{i1}(\omega) - \alpha_{i2}(\omega)) \}}{\Delta(\omega)} \\
b &= \frac{\{ r_{i1}(\omega)(\alpha_{i3}(\omega) + \alpha_{i4}(\omega) - \alpha_{i1}(\omega) - \alpha_{i2}(\omega)) - r_{i1}(\omega)(\alpha_3(\omega) + \alpha_4(\omega) - \alpha_1(\omega) - \alpha_2(\omega)) \}}{\Delta(\omega)}.
\end{align*}
\]

(2.17)

Now

\[
\begin{align*}
\Delta(\omega) &= (2/N)^2 \{ \sum_{p=1}^{N/2} \sin(\omega(2p - 1)) \sum_{q=0}^{N/2-1} \cos(2\omega q) - \\
&\sum_{p=1}^{N/2} \cos(\omega(2p - 1)) \sum_{q=0}^{N/2-1} \sin(2\omega q) \} \\
&= (2/N)^2 \{ \sum_{p=1}^{N/2} \sum_{q=0}^{N/2-1} \sin(\omega(2p - 2q - 1)) \}
\end{align*}
\]

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\[ = (2/N)^2 \left\{ \sum_{p=1}^{N/2} \sum_{q=0}^{N/2-1} \text{imag}(e^{i(\omega(2p-2q-1))}) \right\} \]
\[ = (2/N)^2 \left\{ \text{imag}\left(\sum_{p=1}^{N/2} \sum_{q=0}^{N/2-1} e^{i(\omega(2p-2q-1))}\right) \right\}. \]

We can show that for \( \omega \in (-\pi, 0) \)
\[ \sum_{p=1}^{N/2} \sum_{q=0}^{N/2-1} e^{i(\omega(2p-2q-1))} = i((1 - \cos(\omega N)/2 \sin(\omega))), \]

and for \( \omega = -\pi \)
\[ \sum_{p=1}^{N/2} \sum_{q=0}^{N/2-1} e^{i(\omega(2p-2q-1))} = N/2(N/2 - 1). \]

Therefore, for \( \omega \in (-\pi, 0) \), \( \Delta(\omega) = (2/N)^2(1-\cos(\omega N)/(2 \sin(\omega))) \), and for \( \omega = -\pi \), \( \Delta(\omega) = 0. \)

Thus, for \( \omega \in (-\pi, 0) \) we have
\[ \Delta(\omega) = 0 \iff \cot(\omega N) = 1. \]

Hence
\[ \Delta(\omega) = 0 \iff \omega = 2\pi k/N \text{ for } N/2 + 1 \leq k \leq N - 1. \]

Letting
\[ A = \begin{cases} \omega \in \mathbb{R}_0 & \omega = 2\pi k/N \\ \text{where } N/2 + 1 \leq k \leq N - 1 \end{cases} \]

we conclude that for \( \omega \in \{(-\pi, 0)/A \} \) the system (2.17) has a solution. In practice, we proceed by selecting an \( \omega \) in equation (2.17) and determining the corresponding
values of $a$ and $b$. Observe that the spectrum of $s(n)$ is equal to zero on $A$.

**Example:** We apply the new method *ehilbert* to two examples. First, to a signal from the class of signals encountered earlier $x(n) = [1 \ 2 \ 1 \ 2]$. Then to another signal, the Daubechies scaling filter db16 not belonging to the previous class. We choose constants $a$ and $b$ such that the DTFT is equal to zero at $\omega = -2.4$ and $\omega = -\pi + 0.001$ for $x(n)$ and the filter respectively. The $\omega$ values were selected empirically. Results are compared with the analytic signals obtained using *hilbert*. For both cases we illustrate the results using 256 point DFTs. For $x(n) = [1 \ 2 \ 1 \ 2]$, Figure (2.1) shows the magnitude of the spectra of the analytic signals generated by both methods. As formulated, the spectrum using *ehilbert* is equal to zero at $\omega = -2.4$, and small in the neighborhood of $\omega = -2.4$. This is not the case when using *hilbert*. As expected, the magnitude of the spectrum for the latter case is *symmetric*. For the db16 signal, Figure (2.2) shows the spectrum by both methods. We also note that the magnitude of the spectrum of the DTA signal generated by *ehilbert* vanishes faster than that for the spectrum generated by *hilbert*.

### 2.4 Measuring the effect of *ehilbert*

The degree of aliasing in a wavelet decomposition is determined by a measure of the shiftability of the coefficients. As we know, downsampling in the undecimated wavelet transform decomposition generates aliasing. Downsampling is also a time-varying operation, thus leading to a lack of time-invariance in the wavelet decomposition. Hence, while it is not possible in the undecimated case to have time-invariance, it may be possible to attain a weaker form of time-invariance referred to as shiftability. A transform is defined as shiftable [22] when the coefficient energy in each subband is conserved under input-signal shifts. This is seen to measure the degree or lack thereof, of aliasing generated in a wavelet decomposition. We con-
sider only the signal db16 and the corresponding analytic signals generated using \texttt{hilbert} and \texttt{ehilbert}. Both analytic signal spectra are zero to varying degrees in the region $-(\pi, 0]$. To measure this factor, we measure the degree of aliasing generated when utilized in a wavelet decomposition. Hence the analytic signals generated by the two methods are used as scaling filters in a wavelet decomposition. Quadrature mirror filters associated with each of the two filters served as wavelet filters for the two methods. Following methodology similar to others [5], subband energy for both transforms was determined over 16 circular shifts of an impulse that was fed to the analytic scaling and wavelet filters. Figure (2.3) shows the transform subband energies at the different scales, as a function of input signal shifts.

We observe large oscillations in the subband-energy corresponding to filter generation using \texttt{hilbert} in (b), (c), (d) whereas it is almost constant when using \texttt{ehilbert}. The subband energy in (a) oscillates, but the variation is of order $10^{-8}$. We conclude that the new method generates a spectrum that suppresses more negative frequencies than that obtained using \texttt{hilbert}. Accordingly, aliasing is considerably reduced. Both methods retain the original signal as the real part of the analytic signal. However, as stated earlier, orthogonality of the real and imaginary parts is not maintained by \texttt{ehilbert}.

\section{2.5 Conclusion}

We have proposed a new method \texttt{ehilbert} for generating a DTA signal for which the algorithm \texttt{hilbert} in MATLAB6.5 is a special case ($a = b = 0$). The advantage of the method is that it assures better suppression of negative frequencies. Beside zeroing the DTFT of the DTA signal at negative frequencies $\omega_k = 2\pi k/N$, where $N/2 + 1 \leq k \leq N - 1$, it also zeros the DTFT of the DTA signal at a point in the
Figure 2.1: Discrete analytic signal for $x(n) = [1 \ 2 \ 1 \ 2]$. 

negative frequency range, thus leading to improved shiftability.
Figure 2.2: Discrete analytic signal for Daubechies scaling filter of length 32.
Figure 2.3: Subband energy for both transforms. (a) Level-1 bandpass subband energy, (b) Level-2 bandpass subband energy, (c) Level-3 bandpass subband energy, (d) Level-3 lowpass subband energy.
Chapter 3

New Method: Time domain approach

3.1 Introduction

As opposed to the frequency domain approach where the signal is operated on directly to generate a DTA signal, the time-domain process consists of designing a filter whose output to a real signal is a DTA signal. The concerns here are with the design of a complex “analytic filter” with some desired characteristics such that DTA signals are generated. Qualities desired for the analytic function (the filter or transform output) are invertibility and orthogonality. That is, it should be possible to recover the original real signal from the DTA signal and the real part of the DTA should be orthogonal to its imaginary part. Those desired for the analytic filter are linear phase and real-time implementation. We shall often make reference to all four properties in the context of the filter itself, rather than prescribing them individually to the filter or the DTA signal. How well the analytic filter generates an analytic function in terms of a one-sided transform is measured by the concept of shiftability.
We review the basic time-domain approaches [19], [5], [18] to generating analytic filters. We make brief reference to the standard frequency domain method [13]. In all these cases, it is seen that they lack at least one of the four properties stated earlier. Note that preserving the original signal as the real part allows its recovery. Hence that capability is referred to as preserving invertibility. We present our new method and show that it satisfies all the proprieties stated above. Finally, we make empirical comparison to the method in [5] using the concept of shiftability.

3.2 Basic methods

The method in [18], Sec 10.4.1] details a method for analytic filter design for generating DTA signals. The approach consists of a designing two infinite impulse response (IIR) filters that are in quadrature to generate the real part and the imaginary part of the DTA signal. This approach satisfies the orthogonality condition and real-time capability, but lacks invertibility and linear phase. In [19] and in [5] the procedure consists of the design of a lowpass half-band filter (spectrum in the range \((-\pi/2, +\pi/2\)) and a spectrum shift of \(\pi/2\) to the right. This results in a filter with a one-sided spectrum that is commonly referred to as a complex half-band filter. The practical concern is, of course, to maintain as much attenuation as possible in the negative frequency band \((-\pi, 0)\). In [19] the lowpass filter used was designed using the Parks-McClellan [15] algorithm so as to have generalized linear phase. The resulting analytic signal is invertible, but not orthogonal. In [5] the lowpass filter used is the Daubechies scaling filter. The transform is invertible, but does not have linear phase and lacks orthogonality. We recall that the frequency domain approach [13] consists of transforming the signal into the frequency domain using the DFT, some simple scaling of the positive frequencies and the setting of all negative frequency to zero. The desired DTA is then generated using the IDFT. The problem with this
approach is that the whole signal must be available at once, making it not possible for real-time implementation. Although the transform is invertible and orthogonal, it does not have generalized linear phase.

In this chapter, we propose a new method that possesses all the four desired properties:

- Generalized linear phase.
- Orthogonality.
- Preservation of the original real valued signal.
- Real-time implementation.

In addition, we show reduced shiftability compared to another current technique.

### 3.3 The method

As opposed to the previous procedures where half-band lowpass filters are designed and then complex half-band filters obtained by a $\pi/2$ frequency shift to the right, we consider the design of the complex half-band filter directly. The standard algorithm `hilbert` is used for this purpose. (Recall that we can improve on this using `ehilbert`). The output of this filter generates the corresponding analytic signal. The filter design idea is quite simple: We start with a Kronecker delta function shifted by $N/2$, $\delta(n - N/2)$, or shifted by $(N - 1)/2$, when the length $N$ of the desired filter is even, (odd respectively). The choice of the shift is required to provide generalized linear phase. The delta function has a constant DFT spectrum and a phase $e^{j\pi N/2}$. On application of the `hilbert` to the shifted discrete impulse, we obtain the one-sided or complex half-band filter. The filter preserves positive frequencies and attenuates negative ones. The output of this filter provides the DTA signal.
Figure 3.1: Amplitude spectra for $G(e^{j\omega})$.

The underlying issue in the approximation is the behavior for large $N$. Given that we are approximating the ideal complex half-band filter $G(e^{j\omega})$, Figure 3.1, what are the consequences for $N \to +\infty$?

To resolve this question we first need to determine the closed form expression of the DTFT of the filter which we call discrete time analytic filter (DTAF). After this is accomplished, we address the convergence issue of the DTFT. We provide the proof as $N \to +\infty$ the DTFT converges almost everywhere to the filter with impulse response $g(n)$ and frequency response

$$G(e^{j\omega}) = \begin{cases} 
2, & 0 < \omega < \pi \\
0, & -\pi < \omega < 0. 
\end{cases}$$
3.4 The closed form expression of the DTFT

3.4.1 Case when $N$ is a multiple of 4

Let $x(n)$ be the discrete impulse of length $N$, with $N$ is a multiple of 4, shifted by $N/2$. i.e

$$x(n) = \begin{cases} 
1, & n = N/2 \\
0, & 0 \leq n \leq N - 1, \quad n \neq N/2.
\end{cases}$$

Applying $\text{hilbert}$ to $x(n)$ we obtain the DTA function $z(n)$. Referring to equations (2.1), (2.2), we have for $n$ even

$$z(n) = x(n) + j(2/N) \sum_{p=0}^{N/2-1} x(2p + 1) \cot(\pi(n - (2p + 1))/N)$$

and for $n$ odd

$$z(n) = x(n) + j(2/N) \sum_{p=0}^{N/2-1} x(2p) \cot(\pi(n - 2p)/N).$$

Observe that the only non zero element of $x(n)$ is at $n = N/2$, which is even index, and is equal to 1. therefore

$$z(n) = \begin{cases} 
0, & n \text{ even, and } n \neq N/2 \\
x(N/2), & n = N/2 \\
x(n) + j(2/N)x(2p)\cot(\frac{\pi}{N}n - \frac{\pi}{2}), & n \text{ odd.}
\end{cases}$$
thus

\[
z(n) = \begin{cases} 
0, & n \text{ even, and } n \neq N/2 \\
1, & n = N/2 \\
x(n) + j(2/N)\cot\left(\frac{\pi}{N}n - \frac{\pi}{2}\right), & n \text{ odd.}
\end{cases}
\]

\[
z(n) = \begin{cases} 
0, & n \text{ even} \\
1, & n = N/2 \\
j(2/N)\cot\left(\frac{\pi}{N}n - \frac{\pi}{2}\right), & n \text{ odd.}
\end{cases}
\]

(3.1)

Now we calculate the DTFT \(Z(e^{j\omega})\) of \(z(n)\). We have

\[
Z(e^{j\omega}) = \sum_{n=0}^{N-1} z(n)e^{-j\omega n}
\]

\[
= \sum_{n=0}^{N-1} z(2n+1)e^{-j\omega(2n+1)} + e^{-j\omega N/2}
\]

\[
= \sum_{n=0}^{N-1} z(2n+1)e^{-j\omega(2n+1)} + \sum_{n=N/4}^{N-1} z(2n+1)e^{-j\omega(2n+1)} + e^{-j\omega N/2}.
\]

therefore

\[
Z(e^{j\omega}) = \{(z(1)e^{-j\omega} + z(N-1)e^{-j(N-1)\omega}) + (z(3)e^{-j3\omega} + z(N-3)e^{-j(N-3)\omega}) + \cdots + (z(N/2 - 1)e^{-j(N/2-1)\omega} + z(N/2 - 1)e^{-j(N/2+1)\omega})\} + e^{-j\omega N/2}
\]

\[
= \{(z(1)e^{-j\omega} + z(N-1)e^{-j(N-1)\omega}) + (z(3)e^{-j3\omega} + z(N-3)e^{-j(N-3)\omega}) + \cdots + (z(N/2 - 1)e^{-j(N/2-1)\omega} + z(N/2 - 1)e^{-j(N/2+1)\omega})\} + e^{-j\omega N/2}.
\]

(3.2)

Now we show that for \(1 \leq 2n+1 \leq N/2\), with \(0 \leq n \leq N/2 - 1\)
\[ z(N - (2n + 1)) = -z(2n + 1). \]

Since \((N - (2n + 1))\) is odd, equation (3.1) implies

\[
z(N - (2n + 1)) = j(2/N) \cot\left(\frac{\pi}{N}(N - (2n + 1)) - \frac{\pi}{2}\right) \\
= j(2/N) \cot\left(\pi - \frac{\pi}{N}(2n + 1) - \frac{\pi}{2}\right) \\
= j(2/N) \cot\left(\frac{\pi}{2} - \frac{\pi}{N}(2n + 1)\right) \\
= -j(2/N) \cot\left(\frac{\pi}{N}(2n + 1)\pi/2\right) \\
= -z(2n + 1).
\]

Therefore the equation (3.2) becomes

\[
Z(e^{j\omega}) = \left\{ (z(1)e^{-j\omega} - z(1)e^{-j(N-1)\omega}} + (z(3)e^{-j3\omega} - z(3)e^{-j(N-3)\omega}) + \cdots + \\
(z(\frac{N}{2} - 1)e^{-j(\frac{N}{2}-1)\omega} - z(\frac{N}{2} - 1)e^{-j(N-(\frac{N}{2}-1))\omega}) \right\} + e^{-j\omega N} \\
= \left\{ (z(1)(e^{-j\omega} - e^{-j(N-1)\omega})) + z(3)(e^{-j3\omega} - e^{-j(N-3)\omega}) + \cdots + \\
z(\frac{N}{2} - 1)(e^{-j(\frac{N}{2}-1)\omega} - e^{-j(N-(\frac{N}{2}-1))\omega}) \right\} + e^{-j\omega N}. \\
= e^{-j\omega N/2}\{1 + (z(1)(e^{j\omega(\frac{N}{2}-1)} - e^{-j\omega(\frac{N}{2}-1)})) + z(3)(e^{j\omega(\frac{N}{2}-3)} - e^{-j\omega(\frac{N}{2}-3)}) + \cdots + z(\frac{N}{2} - 1)(e^{j\omega} - e^{-j\omega}) \} \\
= e^{-j\omega N/2}\{1 + 2j z(1)\sin(\omega(\frac{N}{2} - 1)) + 2j z(3)\sin(\omega(\frac{N}{2} - 3)) + \cdots + 2jz(\frac{N}{2} - 1)\sin(\omega) \} \\
= e^{-j\omega N/2}\{1 + 2j \sum_{n=0}^{N-1} z(2n + 1)\sin(\omega(\frac{N}{2} - (2n + 1))) \}.
\]

We substitute for \(z(n + 1)\) its value given by equation (3.1). Hence

\[
Z(e^{j\omega}) = e^{-j\omega N/2}\{1 + 2j \sum_{n=0}^{N-1} j(2/N) \cot\left(\frac{\pi}{N}(2n + 1) - \frac{\pi}{2}\right) \sin(\omega(\frac{N}{2} - (2n + 1))) \}.
\]
\[= e^{-j\omega N/2}\{1 - \frac{4}{N} \sum_{n=0}^{N/2-1} \cot\left(\frac{\pi}{N}(2n + 1) - \frac{\pi}{2}\right) \sin(\omega\frac{N}{2} - (2n + 1))\}\]

\[= e^{-j\omega N/2}\{1 + \frac{4}{N} \sum_{n=0}^{N/2-1} \tan\left(\frac{\pi}{N}(2n + 1)\right) \sin(\omega\frac{N}{2} - (2n + 1))\}\]
thus

\[
  z(n) = \begin{cases} 
    j(2/N) \cot \left( \frac{\pi}{N} n - \frac{\pi}{2} \right), & n \text{ even} \\
    0, & n \text{ odd, and } n \neq N/2 \\
    1, & n = N/2. 
  \end{cases} \quad (3.3)
\]

Now we calculate the DTFT \( Z(e^{j\omega}) \) of \( z(n) \), we have

\[
  Z(e^{j\omega}) = \sum_{n=1}^{N-1} z(n) e^{-j\omega n} \\
  = \sum_{n=1}^{N-1} z(2n) e^{-j\omega 2n} + e^{-j\omega N/2} \\
  \quad \text{for } n = 1, 2, \ldots, \frac{N}{2}-1 \\
  = \sum_{n=1}^{\frac{N}{2}-1} z(2n) e^{-j\omega 2n} + \sum_{n=1/2(N/2+1)}^{\frac{N}{2}-1} z(2n) e^{-j\omega 2n} + e^{-j\omega N/2}.
\]

therefore

\[
  Z(e^{j\omega}) = \{(z(2)e^{-j2\omega} + z(N - 2)e^{-j(N-2)\omega}) + (z(4)e^{-j4\omega} + z(N - 4)e^{-j(N-4)\omega}) + \cdots + (z(N/2 - 1)e^{-j(N/2-1)\omega} + z(\frac{N}{2}+1)e^{-j(N/2+1)\omega}) \} + e^{-j\omega N/2}
\]

\[
  = \{(z(2)e^{-j2\omega} + z(N - 2)e^{-j(N-2)\omega}) + (z(4)e^{-j4\omega} + z(N - 4)e^{-j(N-4)\omega}) + \cdots +
  (z(N/2 - 1)e^{-j(N/2-1)\omega} + z(\frac{N}{2}+1)e^{-j(N/2+1)\omega}) \} + e^{-j\omega N/2}.
\] \quad (3.4)

Now we show the formula \( z(N - 2n) = -z(2n) \), for \( 2 \leq 2n \leq N/2 - 1 \) and \( 0 \leq n \leq N/2 - 1 \). We have \( N - 2n \) is even, thus equation (3.3) implies

\[
  z(N - 2n) = j(2/N) \cot \left( \frac{\pi}{N} (N - 2n) - \frac{\pi}{2} \right) \\
  = j(2/N) \cot \left( \pi - \frac{\pi}{N} 2n - \frac{\pi}{2} \right) \\
  = j(2/N) \cot \left( \pi/2 - \frac{\pi}{N} 2n \right) \\
  = -j(2/N) \cot \left( \frac{\pi}{N} 2n - \pi/2 \right)
\]
Therefore the equation (3.4) become

\[
Z(e^{j\omega}) = \{(z(2)e^{-j2\omega} - z(2)e^{-j(N-2)\omega}) + (z(4)e^{-j4\omega} - z(4)e^{-j(N-4)\omega}) + \cdots +
\]
\[
(z(\frac{N}{2} - 1)e^{-j(\frac{N}{2} - 1)\omega} - z(\frac{N}{2} - 1)e^{-j(N-(\frac{N}{2} - 1))\omega})\} + e^{-j\omega N/2}
\]
\[
= \{(z(2)(e^{-j2\omega} - e^{-j(N-2)\omega})) + (z(4)(e^{-j4\omega} - e^{-j(N-4)\omega}) + \cdots +
\]
\[
z(\frac{N}{2} - 1)(e^{-j(\frac{N}{2} - 1)\omega} - e^{-j(N-(\frac{N}{2} - 1))\omega})\} + e^{-j\omega N/2}
\]
\[
= e^{-j\omega N/2}\{1 + (z(2)(e^{j\omega(\frac{N}{2} - 2)} - e^{-j\omega(\frac{N}{2} - 2)})) + (z(4)(e^{j\omega(\frac{N}{2} - 4)} -
\]
\[
e^{-j\omega(\frac{N}{2} - 4)}) + \cdots + z(\frac{N}{2} - 1)(e^{j\omega} - e^{-j\omega})\}
\]
\[
= e^{-j\omega N/2}\{1 + 2jz(1)\sin(\omega(\frac{N}{2} - 2)) + 2jz(4)\sin(\omega(\frac{N}{2} - 4))
\]
\[
+ \cdots + 2jz(\frac{N}{2} - 1)\sin(\omega)\}
\]
\[
= e^{-j\omega N/2}\{1 + 2j \sum_{n=1}^{\frac{1}{2}(\frac{N}{2} - 1)} z(2n)\sin(\omega(\frac{N}{2} - 2n))\}.
\]

We substitute \(z(2n)\) by its value given by equation (3.3) we get

\[
Z(e^{j\omega}) = e^{-j\omega N/2}\{1 + 2j \sum_{n=1}^{\frac{1}{2}(\frac{N}{2} - 1)} j(2/N)\cot(\frac{\pi}{N}2n - \frac{\pi}{2})\sin(\omega(\frac{N}{2} - 2n))\}
\]
\[
= e^{-j\omega N/2}\{1 - 4 \sum_{n=1}^{\frac{1}{2}(\frac{N}{2} - 1)} \cot(\frac{\pi}{N}2n - \frac{\pi}{2})\sin(\omega(\frac{N}{2} - (2n + 1)))\}
\]
\[
= e^{-j\omega N/2}\{1 + 4 \sum_{n=1}^{\frac{1}{2}(\frac{N}{2} - 1)} \tan(\frac{\pi}{N}2n)\sin(\omega(\frac{N}{2} - 2n))\}.
\]

### 3.4.3 Case when \(N\) is odd

We first need to find the explicit formula of the DTA signal corresponding to a real signal for the case \(N\) odd using the algorithm in [7] (algorithm not given in [13]). In [7], for a real discrete signal of odd length \(N\), the procedure for deriving its discrete
analytic signal consists of three steps:

- Compute the $N$-point DFT of $x(n)$.
- Form the $N$-point DFT of the discrete analytic signal by multiplying the $N$-point DFT of $x(n)$ by the vector:

\[
a(n) = \begin{cases} 
1, & n = 0. \\
2, & 1 \leq n \leq \frac{N-1}{2} \\
0, & \frac{N+1}{2} \leq n \leq N - 1.
\end{cases}
\]

Thus, the $N$-point DFT of the discrete analytic signal is,

\[
Z(k) = \begin{cases} 
X(k), & k = 0. \\
2X(k), & 1 \leq k \leq \frac{N-1}{2} \\
0, & \frac{N+1}{2} \leq k \leq N - 1.
\end{cases}
\]

- Obtain the discrete analytic signal by computing the inverse DFT of the $N$-point DFT:

\[
z(n) = \frac{1}{N} \sum_{k=0}^{N-1} Z(k)e^{j2\pi kn/N}
\]

Thus,

\[
z(n) = 1/N \sum_{k=0}^{N-1} Z(k)e^{j2\pi kn/N} = 1/N \sum_{k=0}^{N-1} Z(k)e^{j2\pi kn/N} = \frac{1}{N}X(0) + 2/N \sum_{k=1}^{N/2} X(k)e^{j2\pi kn/N}
\]
\[
\frac{1}{N} X(0) + 2/N \sum_{k=1}^{N-1} X(k)e^{j2\pi kn/N} x(n) - x(n) = \frac{1}{N} x(n) + 2/N \sum_{k=1}^{N-1} X(k)e^{j2\pi kn/N} - 1/N \sum_{k=0}^{N-1} X(k)e^{j2\pi kn/N} \\
= x(n) + 1/N \sum_{k=1}^{N-1} X(k)e^{j2\pi kn/N} - 1/N \sum_{k=\frac{N-1}{2}}^{N-1} X(k)e^{j2\pi kn/N}.
\]

Changing variable \( p = k - \frac{N-1}{2} \),

\[
\sum_{k=\frac{N-1}{2}}^{N-1} X(k)e^{j2\pi kn/N} = \sum_{k=1}^{N-1} X(k + \frac{N-1}{2})e^{j2\pi (k-1/2)n/N} \\
= \sum_{k=1}^{N-1} (-1)^n X(k + \frac{N-1}{2})e^{j2\pi (k-1/2)n/N}.
\]

Therefore

\[
z(n) = x(n) + 1/N \sum_{k=1}^{N-1} X(k)e^{j2\pi kn/N} - \sum_{k=1}^{N-1} (-1)^n X(k + \frac{N-1}{2})e^{j2\pi (k-1/2)n/N} \\
= x(n) + 1/N \sum_{k=1}^{N-1} \{X(k) - (-1)^n X(k + \frac{N-1}{2})e^{-j\pi n/N}\}e^{-j2\pi kn/N}.
\]

Also, since

\[
X(k + \frac{N-1}{2}) = \sum_{p=0}^{N-1} x(p)e^{-j2\pi p(k+\frac{N-1}{2})/N} \\
= \sum_{p=0}^{N-1} x(p)(-1)^p e^{-j2\pi pk/N} e^{j\pi p/N}
\]

therefore

\[
X(k) - (-1)^n X(k + \frac{N-1}{2})e^{-j\pi n/N} = \sum_{p=0}^{N-1} x(p)(1 - (-1)^p e^{j\pi (p-n)/N})e^{-j2\pi pk/N}.
\]

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Thus

\[
z(n) = x(n) + 1/N \sum_{k=1}^{N-1} \left\{ \sum_{p=0}^{N-1} x(p)(1 - (-1)^{p+n}e^{j\pi(p-n)/N})e^{-j2\pi nk/N} \right\}e^{-j2\pi kn/N}
\]

\[
= x(n) + 1/N\left\{ \sum_{p=0}^{N-1} x(p)(1 - (-1)^{p+n}e^{j\pi(p-n)/N})\right\}\left(\sum_{k=1}^{N-1} e^{j2\pi k(n-p)/N}\right).
\]

(3.5)

Observe that for \( p = n \) we get \( \sum_{p=0}^{N-1} x(p)(1 - (-1)^{p+n}e^{j\pi(p-n)/N}) = 0 \). Therefore equation (3.5) becomes

\[
z(n) = x(n) + 1/N\left\{ \sum_{p=0, p \neq n}^{N-1} x(p)(1 - (-1)^{p+n}e^{j\pi(p-n)/N})\right\}\left(\sum_{k=1}^{N-1} e^{j2\pi k(n-p)/N}\right).
\]

Now we evaluate \( \sum_{k=1}^{N-1} e^{j2\pi k(n-p)/N} \) for \( p \neq n \). We get

\[
\sum_{k=1}^{N-1} e^{j2\pi k(n-p)/N} = \frac{(-1)^{n+p} - e^{j\pi(n-p)/N}}{2j\sin(\pi(n-p)/N)}.
\]

Thus

\[
z(n) = x(n) + 1/N \sum_{p=0}^{N-1} x(p)\frac{-e^{j\pi(n-p)} + (-1)^{n+p} + (-1)^{n+p} - e^{-j\pi(n-p)/N}}{2j\sin(\pi(n-p)/N)}.
\]

\[
= x(n) + j/N \sum_{p=0, p \neq n}^{N-1} x(p)\frac{\cos(\pi(n-p)/N) - (-1)^{n+p}}{\sin(\pi(n-p)/N)}.
\]

(3.6)

Now, let \( x(n) \) be the discrete impulse of length \( N \), with \( N \) odd, shifted by \( \frac{N-1}{2} \). That is

\[
x(n) = \begin{cases} 
1, & n = \frac{N-1}{2} \\
0, & 0 \leq n \leq N-1, \ n \neq \frac{N-1}{2}.
\end{cases}
\]
Applying the algorithm [7] to \( x(n) \) to get the DTA denoted by \( z(n) \), equation (3.6) implies,

\[
z(n) = x(n) + j/N \sum_{p=0}^{N-1} x(p) \frac{\cos(\pi(n - p)/N) - (-1)^{n+p}}{\sin(\pi(n - p)/N)}.
\]

Therefore for \( n = \frac{N-1}{2} \), \( z(n) = 1 \), and for \( n \neq \frac{N-1}{2} \) we have,

\[
z(n) = x(n) + j/N \frac{x(N-1)/2 \cos(\pi(n - \frac{N-1}{2})/N) - (-1)^n}{\sin(\pi(n - \frac{N-1}{2})/N)}
\]

\[
= x(n) + j/N \frac{\cos(\pi(n - \frac{N-1}{2})/N) - (-1)^n}{\sin(\pi(n - \frac{N-1}{2})/N)}
\]

\[
= j/N \frac{\cos(\pi(n - \frac{N-1}{2})/N) - (-1)^n}{\sin(\pi(n - \frac{N-1}{2})/N)}.
\]

Now evaluate the DTFT of \( z(n) \) denoted by \( Z(e^{jwt}) \), we have

\[
Z(e^{jwt}) = \sum_{n=0}^{N-1} z(n)e^{-jwn}
\]

\[
= \sum_{n=0}^{N-1} z(2n)e^{-jwn} + \sum_{n=0}^{N-3} z(2n+1)e^{-jwn(2n+1)}
\]

\[
= (z(0) + z(N-1)e^{-jw(N-1)}) + (z(1)e^{-jw} + z(n-2)e^{-jw(N-2)}) + \cdots +
\]

\[
(z(\frac{N-1}{2} - 1)e^{-jw(\frac{N-1}{2} - 1)}) + z(\frac{N-1}{2} + 1)e^{-jw(\frac{N-1}{2} + 1)} + e^{-jw(\frac{N-1}{2})}.
\]

Now we show that for \( 0 \leq p \leq N - 1 \) we have \( z(N - 1 + p) = -z(p) \). For \( p \neq \frac{N-1}{2} \),

\[
z(N - 1 + p) = j/N \frac{\cos(\pi(N - 1 - p - \frac{N-1}{2})/N) - (-1)^{N-1-p}}{\sin(\pi(N - 1 - p - \frac{N-1}{2})/N)}
\]

\[
= j/N \frac{\cos(\pi\frac{N-1}{2} - p)/N) - (-1)^p}{\sin(\pi\frac{N-1}{2} - p)/N)}
\]

\[
= j/N \frac{\cos(\pi(p - \frac{N-1}{2})/N) - (-1)^p}{\sin(\pi(p - \frac{N-1}{2})/N)}
\]

\[
= -z(p).
\]

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Therefore

\[
Z(e^{j\omega}) = (z(0) - z(0)e^{-j\omega(N-1)}) + (z(1)e^{-j\omega} - z(1)e^{-j\omega(N-2)}) + \cdots + \\
(z\left(\frac{N-1}{2} - 1\right)e^{-j\omega\left(\frac{N-1}{2}\right)} - z\left(\frac{N-1}{2} - 1\right)e^{-j\omega\left(\frac{N-1}{2} + 1\right)} + e^{-j\omega\frac{N-1}{2}})
\]

\[
e^{-j\omega\frac{N-1}{2}}\left\{1 + z(0)(e^{j\omega\frac{N-1}{2}} - e^{-j\omega\frac{N-1}{2}}) + \cdots + z\left(\frac{N-1}{2} - 1\right)(e^{j\omega} - e^{-j\omega})\right\}
\]

\[
e^{-j\omega\frac{N-1}{2}}\left\{1 + z(0)2j\sin\left(\frac{N-1}{2}\omega\right) + \cdots + z\left(\frac{N-1}{2} - 1\right)2j\sin(\omega)\right\}
\]

\[
e^{-j\omega\frac{N-1}{2}}\left\{1 + \sum_{k=0}^{\frac{N-1}{2}-1} z(k)2j\sin\left(\frac{N-1}{2} - k\right)\omega)\right\}
\]

(3.8)

Using equation (3.7) in equation (3.8), we get

\[
Z(e^{j\omega}) = e^{-j\omega\frac{N-1}{2}}\left\{1 + \sum_{k=0}^{\frac{N-1}{2}-1} \frac{j/N}{\sin(\pi(k - \frac{N-1}{2}))/N} \cos(\pi(k - \frac{N-1}{2}))/N) - (-1)^k2j\sin\left(\frac{N-1}{2} - k\right)\omega)\right\}
\]

\[
e^{-j\omega\frac{N-1}{2}}\left\{1 - 2/N\sum_{k=0}^{\frac{N-1}{2}-1} \frac{\cos(\pi(k - \frac{N-1}{2}))/N) - (-1)^k}{\sin(\pi(k - \frac{N-1}{2}))/N) \sin\left(\frac{N-1}{2} - k\right)\omega)\right\}
\]

Letting \(k = \frac{N-1}{2} - p\), we get

\[
Z(e^{j\omega}) = e^{-j\omega\frac{N-1}{2}}\left\{1 + 2/N\sum_{p=1}^{\frac{N-1}{2}} \frac{\cos(\frac{\pi p}{N}) - (-1)^p}{\sin(\frac{\pi p}{N})} \sin(p\omega)\right\}
\]

Hence we finally conclude that

\[
Z(e^{j\omega}) = \begin{cases} 
  e^{-j\omega N/2}\left\{1 + \frac{1}{N}\sum_{n=0}^{N-1} \tan\left(\frac{\pi}{N}(2n + 1)\right) \sin(\omega\left(\frac{N}{2} - (2n + 1)\right))\right\}, & N \text{ multiple of } 4 \\
  e^{-j\omega N/2}\left\{1 + \frac{1}{N}\sum_{n=0}^{N-1} \tan\left(\frac{\pi}{N}(2n)\right) \sin(\omega\left(\frac{N}{2} - 2n\right))\right\}, & N \text{ even } N/2 \text{ odd} \\
  e^{-j\omega\frac{N-1}{2}}\left\{1 + 2/N\sum_{p=1}^{\frac{N-1}{2}} \frac{\cos(\frac{\pi p}{N}) - (-1)^p}{\sin(\frac{\pi p}{N})} \sin(p\omega)\right\}, & N \text{ odd}.
\end{cases}
\]

(3.9)
3.5 Convergence of the DTFT

Having found the closed form expression of the frequency response of the DTAF, we show that it’s DTFT converge almost everywhere to the filter with impulse response $g(n)$ whose spectra is plotted in figure (3.1), and its frequency transform is as follows:

$$G(e^{j\omega}) = \begin{cases} 
2, & 0 < \omega < \pi \\
0, & -\pi < \omega < 0.
\end{cases} \quad (3.10)$$

The proof is shown for the case $N$ is multiple of 4. The $\{N \text{ even} \text{ and } N/2 \text{ odd}\}$ and $\{N \text{ odd}\}$ cases are still being investigated.

**Theorem 1:** The function

$$\varphi(e^{j\omega}) = 1 + \frac{4}{N} \sum_{n=0}^{N-1} \tan\left(\frac{\pi}{N}(2n+1)\right) \sin(\omega \left(\frac{N}{2} - (2n+1)\right))$$

converges almost everywhere to $G(e^{j\omega})$ given by equation (3.10), with $N = 2^p$, when $N \to +\infty$. **Proof:**

$$\frac{4}{N} \sum_{n=0}^{N-1} \tan\left(\frac{\pi}{N}(2n+1)\right) \sin(\omega \left(\frac{N}{2} - (2n+1)\right)) \right\} = \frac{4}{N} \sum_{n=0}^{N-1} \cot\left(\frac{\pi}{N}(2n+1)\right) \sin(\omega (2n+1)) \right\}.$$ 

We also have

---

1 Proof due to Prof. J. Michael Wilson, UVM Department of Mathematics and Statistics
\[ \frac{4}{N} \sum_{n=0}^{N-1} \cot \left( \frac{\pi}{N}(2n + 1) \right) \sin(\omega(2n + 1)) \right] = \frac{4}{N} \sum_{n=0}^{N-1} \left( \cot \left( \frac{\pi}{N}(2n + 1) \right) - \frac{N}{(2n + 1)\pi} \right) \sin(\omega(2n + 1)) + \frac{4}{\pi} \sum_{n=0}^{N-1} \frac{\sin((2n + 1)\pi)}{2n + 1}. \] (3.11)

The second term \( \frac{4}{\pi} \sum_{n=0}^{N-1} \frac{\sin((2n + 1)\pi)}{2n + 1} \) converges to 1 on \((0, \pi)\). We claim that the first sum \( \frac{4}{N} \sum_{n=0}^{N-1} \left( \cot \left( \frac{\pi}{N}(2n + 1) \right) - \frac{N}{(2n + 1)\pi} \right) \sin(\omega(2n + 1)) \) converges to 0 when \( N \) converges to \(+\infty\) with \( N = 2^p \), \( p \) positive integer on \((\alpha, \pi)\).

**Lemma:** For \( 0 < \omega < \pi \), \(| \cot(\omega) - \frac{1}{\omega} | \leq c \omega \), where, \( c \) is a positive real number.

**Proof:** Assume \( \omega \) very small, then \(| \cot(\omega) - \frac{1}{\omega} | = \left| \frac{\omega \cos(\omega) - \sin(\omega)}{\omega \sin(\omega)} \right| \). A power series argument implies \( |\omega \sin(\omega)| > c\omega^2 \), while \( |\omega \cos(\omega) - \sin(\omega)| < c'\omega^3 \). Thus \( |\cot(\omega) - \frac{1}{\omega}| \leq c \omega \), where \( c \) is a positive real number.

Now the lemma implies that \( |\cot \left( \frac{\pi(2n + 1)}{N} \right) - \frac{N}{\pi(2n + 1)}| \leq \frac{c(2n + 1)}{N} \).

Denote by \( c_{n,N} = \cot \left( \frac{\pi(2n + 1)}{N} \right) - \frac{N}{\pi(2n + 1)} \), and by \( \psi_N = \sum_{n=0}^{N-1} c_{n,N} \sin((2n + 1)\omega) \).

Thus \( |c_{n,N}| \leq \frac{c(2n + 1)}{N^2} \).

Therefore

\[
\int_0^{2\pi} |\psi(\omega)|^2 d\omega \leq c \sum_{n=0}^{N-1} \left( \frac{2n + 1}{N^2} \right)^2 \\
\leq \frac{c}{N^2} \sum_{n=0}^{N-1} (2n + 1)^2 \\
\leq C/N.
\]

where \( C \) is a real positive number.

Let \( \epsilon > 0 \) and \( 2^{-l/3} \leq \epsilon \). If \( k \leq l \) and \( |\psi_N_k(\omega)| > \epsilon \), then \( |\psi_N_k(\omega)| > \epsilon \geq 2^{-l/3} \geq \]
\[ 2^{-k/3} \leq \epsilon \] Thus, \( \{ \omega \in (0, \pi) : |\psi_{N,k}(\omega)| > \epsilon \} \leq C 2^{-k/3} \). Thus

\[
\left\{ \omega \in [-\pi, \pi] : \sup_{k \geq l} |\psi_{N,k}(\omega)| > \epsilon \right\} \leq C \sum_{k \geq l} 2^{-k/3} \leq C 2^{-l/3}.
\]

because \( C 2^{-l/3} \) converges to 0 when \( l \) converges to \(+\infty\), we have \( \lim_{N \to \infty} \psi(\omega) = 0 \) almost everywhere on \([-\pi, \pi]\). Therefore \( 1 + \frac{4}{N} \sum_{n=0}^{N-1} \tan \left( \frac{\pi}{N}(2n+1) \right) \sin(\omega(\frac{N}{2} - (2n+1))) \} \) converges to 2 on \((0, \pi)\). The convergence almost everywhere of \( 1 + \frac{4}{N} \sum_{n=0}^{N-1} \tan \left( \frac{\pi}{N}(2n+1) \right) \sin(\omega(\frac{N}{2} - (2n+1))) \} \) to 0 on \((-\pi, 0)\) is similar.

Therefore, we can conclude that \( 1 + \frac{4}{N} \sum_{n=0}^{N-1} \tan \left( \frac{\pi}{N}(2n+1) \right) \sin(\omega(\frac{N}{2} - (2n+1))) \} \) converges almost everywhere to \( G(e^{j\omega}) \) given by equation (3.10). Hence the DTAF will attenuate the negative frequencies, and preserve the positive frequencies of an input signal.

### 3.6 Properties of the DTAF

#### 3.6.1 Generalized linear phase

**Theorem 3.2:** The DTAF has generalized linear phase.

**Proof:** The proof follows from equation (3.9).

#### 3.6.2 Orthogonality

**Theorem 3.3:** The imaginary part and real part of the DTAF are orthogonal.

**Proof:** The proof is based on Parseval’s relation

\[
\sum_{n=-\infty}^{+\infty} g(n)h(n)^* = \frac{1}{2\pi} \int_{-\pi}^{\pi} G(e^{j\omega})H(e^{j\omega})^* d\omega. \tag{3.12}
\]
where \( g(n) \) and \( h(n) \) are two real valued sequence. We show orthogonality for the case \( N \) multiple of 4. Denote by \( r(n), i(n) \) the real part and the imaginary part of the time domain expression of the DTAF \( z(n) \) respectively. Thus, \( z(n) = r(n) + ji(n) \). Therefore we have

\[
r(n) = \begin{cases} 
1, & n = N/2 \\
0, & 0 \leq n \leq N - 1, \text{ and } n \neq N/2.
\end{cases}
\]

Thus the DTFT denoted by \( R(e^{j\omega}) \) of \( r(n) \) is \( R(e^{j\omega}) = e^{-j\omega N/2} \). Denote by \( I(e^{j\omega}) \) the DTFT of \( i(n) \). Thus

\[
I(e^{j\omega}) = -j(Z(e^{j\omega}) - R(e^{j\omega}))
\]

Using the equation (3.9) we have \( Z(e^{j\omega}) = e^{-j\omega N/2}\left\{1 + \frac{4}{N} \sum_{n=0}^{N-1} \tan\left(\frac{\pi}{N}(2n + 1)\right) \sin(\omega(\frac{N}{2} - (2n + 1)))\right\} \), therefore

\[
I(e^{j\omega}) = -j\left\{e^{-j\omega N/2}\left\{1 + \frac{4}{N} \sum_{n=0}^{N-1} \tan\left(\frac{\pi}{N}(2n + 1)\right) \sin(\omega(\frac{N}{2} - (2n + 1)))\right\} - e^{-j\omega N/2}\right\}
\]

\[
= -je^{-j\omega N/2}\left\{\frac{4}{N} \sum_{n=0}^{N-1} \tan\left(\frac{\pi}{N}(2n + 1)\right) \sin(\omega(\frac{N}{2} - (2n + 1)))\right\}.
\]

Let \( c(n) \) be any real valued sequence with its DTFT denoted by \( C(e^{j\omega}) \). Therefore

\[
c(n) * z(n) = c(n) * r(n) + jc(n) * i(n).
\]

Therefore the real part of \( c(n)*z(n) \) is \( c(n)*r(n) \) and the imaginary part is \( c(n)*i(n) \). Denote by

\[
\Delta(e^{j\omega}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} C(e^{j\omega})I(e^{j\omega})\{C(e^{j\omega})R(e^{j\omega})\}^* d\omega
\]
Thus

\[
\Delta(e^{j\omega}) = \frac{-j}{2\pi} \int_{-\pi}^{\pi} |C(e^{j\omega})|^2 e^{-j\omega N/2}, e^{j\omega N/2} \left\{ \frac{4}{N} \sum_{n=0}^{N-1} \tan\left(\frac{\pi}{N} (2n + 1)\right) \left(\omega - \frac{N}{2} - (2n + 1)\right) \right\}^2 d\omega
\]

\[
= \frac{-j}{2\pi} \int_{-\pi}^{\pi} |C(e^{j\omega})|^2 \left\{ \frac{4}{N} \sum_{n=0}^{N-1} \tan\left(\frac{\pi}{N} (2n + 1)\right) \sin\left(\omega - \frac{N}{2} - (2n + 1)\right) \right\}^2 d\omega.
\]

(3.13)

Observe that the integrand in equation (3.13) is periodic of period $2\pi$ and is odd function. Therefore the value of the integral given by equation (3.13) is zero. Thus $\Delta(e^{j\omega}) = 0$. Now we use the Parseval’s relation given by equation (3.12), thus

\[
\sum_{n=-\infty}^{+\infty} (c(n) \ast r(n))(c(n) \ast i(n))^* = \frac{1}{2\pi} \int_{-\pi}^{\pi} C(e^{j\omega}) I(e^{j\omega}) \left\{ C(e^{j\omega}) R(e^{j\omega}) \right\}^* d\omega
\]

\[
= \Delta(e^{j\omega})
\]

\[
= 0.
\]

Therefore the real part and imaginary part of the DTAF are orthogonal for the case $N$ is multiple of 4. Similar method used for proving the orthogonality for the cases \{ $N$ even and $N/2$ odd $\} \text{ and } \{ N$ odd $\}.$

3.6.3 Approximate shiftability

Theorem 3.4: The DTAF realizes approximate shiftability.

Proof: Using Theorem (3.1) we can conclude that because the DTAF attenuates the negative frequencies and preserves the positive frequencies, then it has approximate shiftability.
3.6.4 Current methods and comparison

In [18], Sec 10.4.1, the authors proposed a method to generate a DTA. This approach consists of a design of two IIR filters which are in quadrature to generate the real part and the imaginary part of the DTA. Although this approach satisfies the orthogonality condition, it lacks invertibility. In [19], and in [5] the proposed methods consist of a lowpass filter design and a spectrum shift to the right, maintaining optimal attenuation in the negative frequency band. In [19], the lowpass filter used was designed using the Parks-McClellan algorithm in order to have generalized linear phase. This transform is invertible and has general linear phase, however it’s not orthogonal. In [5] the lowpass filter used is the Daubechies scaling filter. This transform is invertible, but does not have generalized linear phase and lacks orthogonality. The frequency domain approach used in [13] is invertible and orthogonal, however does not have general linear phase and it’s not a real-time approach. Our method is invertible, has linear phase, is orthogonal, and is implementable in real-time.

3.7 Experimental results

The method described above was used to generate a DTAF filter for \( N = 16 \). Results are compared with the DTA signal obtained using the Daubechies scaling filter of length 16 shifted by \( \pi/2 \) used in [5]. For both cases we illustrate the results using 256 point DFTs.

Figure (3.2) shows the magnitude of the spectra of the DTAF filter and db8 scaling filter shifted by \( \pi/2 \). Figure (3.2) shows that both filters attenuate negative frequencies and preserve the positive ones. However we see that the DTAF has greater attenuation compared to the db8 lowpass filter shifted by \( \pi/2 \). In order to appreciate this last statement, we conduct the following experiment. The degree of aliasing in a wavelet decomposition is determined by a measure of the shiftability
Figure 3.2: Amplitude spectra for lowpass filter shifted by $\pi/2$ and for DTAF.

Figure 3.3: Comparison of shiftability for both methods. (a) Level-1 highpass subband-energy, (b) Level-2 highpass subband-energy, (c) Level-3 highpass subband-energy, (d) Level-3 lowpass subband-energy.
of the coefficients. Subband energy for both transforms, using db8 scaling filter shifted by $\pi/2$ and DTAF of length 16, was determined over 16 circular shifts of an impulse that was fed to the both filters DTAF and db8 scaling filter shifted by $\pi/2$ which precede the filter bank. Figure (3.3) shows the transform subband energies at the different scales, as a function of input signal shifts. In Figure (3.3) we observe large oscillations in the subband-energy corresponding to the transform using the db8 shifted by $\pi/2$ comparing to the transform using the DTAF at level-3 highpass and level-3 lowpass subband-energy. We can conclude then aliasing is reduced using the DTAF when compared to the db8 filter shifted by $\pi/2$. We have compared our method to that in [5] since in [[5], Sec.3.3, Proposition 1] the author has shown that using Daubechie’s scaling filter of length 16 shifted by $\pi/2$, the DTA generated by his method has approximate shiftability.

We can reduce shiftability at the cost of orthogonality and linear phase by applying 

 philosophically in the generation of the filter. That is, apply it to the Kronecker impulse shifted by $N/2$, or $(N-1)/2$ when the length of the desired filter is even, (odd respectively). Figure (3.4) shows the magnitude of the spectra of the ehilbert filter of length 16 and that of the db8 scaling filter shifted by $\pi/2$. We observe greater attenuation in the low negative frequencies when compared with the DTAF in Figure (3.2). This fact will ensure a reduction of aliasing in the level-1 high pass. Figure (3.5) shows the transform subband energies at the different scales, as a function of input signal shifts. We observe in figure (3.5) considerable reduction of the shiftability in the highpass filter at the first level.

### 3.8 Summary and conclusions

In this chapter, we have reviewed methods [19], [5], [18], [13] currently employed for generating a DTA signal. We have shown that they lack at least one of the following
Figure 3.4: Amplitude spectra for lowpass filter shifted by $\pi/2$ and for hilbert applied to allpass.

Figure 3.5: Comparison of shiftability for both methods. (a) Level-1 highpass subband-energy, (b) Level-2 highpass subband-energy, (c) Level-3 highpass subband-energy, (d) Level-3 lowpass subband-energy.
properties: generalized linear phase, orthogonality, preservation of the original real valued signal and real-time implementation. We have shown that the new method satisfies all the proprieties stated above. We compared our method experimentally to the method used in [5], and observe that our method improves shiftability to a greater degree than that in [5].
Chapter 4

Conclusion

This study concerns methods for generation of a discrete analytic signal. The main contribution of this thesis is the proposition of two methods for generation of a discrete analytic signal corresponding to a real valued sequence. These two methods have advantages in comparison to the current methods.

4.1 Summary of work

The main contribution of this thesis is the introduction of two methods. The first method, which is a frequency domain approach, solves the defect in [13] which is the failure to generate a DTA for specific signal in addition to realization of approximate shiftability. The second method, which is a time domain approach, is a method that generates a DTA with satisfaction of the following properties: generalized linear phase, orthogonality, preservation of the original signal, and realization of approximate shiftability. Current methods do not satisfy at least one of these properties.

In Chapter 2, we considered a frequency domain approach, [13] which consists of transforming the signal into frequency domain using the DFT, then set the negative
frequency to zero. The desired DTA is then generated by using the IDFT.

We show that this method fails to generate a DTA for specific discrete-time signals. An alternative method based on the DTFT spectrum is developed which solves the defect. Both methods have the same redundancy. The new analytic signal preserves the original signal (real part). We prove also that the new method preserves the zeros of its discrete spectrum in the negative frequencies. The two techniques are compared for the degree of aliasing generated by measuring their shiftability, [22]. The advantage of the new method is in the introduction of one additional zero of the continuous spectrum of the original signal at a negative frequency, and a corresponding approximate shiftability. The approximate shiftability by the new method compared to the method in [13] was shown empirically.

In Chapter 3, we have reviewed the current methods [19], [5], [18], [13], for generating a DTA and showed that these methods lack at least one of the following properties: generalized linear phase, orthogonality, preservation of the original real valued signal, and a realization of approximate in shiftability. We proposed a new method for generating the DTA. We have shown that the new method satisfies all the proprieties stated above, and it is an appropriate method for real time applications. We compared our method experimentally to the method used in [5], and observed that our method improves shiftability better than the method in [5]. To our knowledge there are no methods that satisfy all of these properties, other than the new method that we proved that it satisfies the properties stated above.

4.1.1 Suggestions for further research

We conclude this chapter by suggesting promising directions for future research. The advantage of processing DTA signals instead of the original signal is seen in many applications. In [10], [14] we see application to feature detection. Features
are formed when the phase values of the Fourier components at any point are maximally close, the closeness being measured by the so-called phase congruency [16] [10]. The amplitude of the DTA function, referred to as local energy, is seen to be proportional to Phase Congruency. In [14] the authors extended the concept of DTA function to multi-resolution analytic (MRA) functions using the spectral representation of a wreath product cyclic group. They applied this novel concept to feature detection problems. In [5], [9], [21], we see application of DTA signals to the reduction in shift sensitivity in 1-D discrete wavelet transforms (DWT). In 2-D DWT we see an improvement in directionality. The DWT is well recognized as an efficient tool in signal processing. However its shift sensitivity, poor directionality and implicit phase information, undermine its usage in many applications. Hence the introduction of a Complex Wavelet Transform [5]. In [5], Fernandes showed that processing a DTA instead of the original signal reduces shift sensitivity, and in image processing it improves directionality. In [5] it is shown that the complex wavelet transform is generated by the usual filter bank implementation, preceded by a lowpass filter shifted by $\pi/2$. In [9], Kingsbury developed the dual-tree wavelet transform which consists of a quadrature pair of discrete wavelet transforms. In [21], the author generalizes the spectral factorization method to design a quadrature wavelet pair with specified length and vanishing moment multiplicity. In [3], we see application to texture segmentation. In [3], Bulow et al. extended the concept of 1-D analytic signal to multidimensional signals by developing the hypercomplex signal and using the phase in texture segmentation. In [17], we see signal extraction from noisy data. In [17], the authors developed a new method for estimating a signal in noise called analytic stationary wavelet transform thresholding, where, using the Discrete Hilbert Transform, they create a complex-valued wavelet coefficient from which an amplitude vector is defined. In [12] Lebold et al. we see an application in vibration analysis. The authors, discussed several methods for vibration analysis in
which some methods use the Hilbert Transform to estimate the envelope of a band
passed segment of the time synchronous averaged signal. In [20], we see application
in single-side-band transmission. Transmission of the entire spectrum represents a
waste of frequency space and a waste of power in the carrier. This has resulted in
the widespread use of the single-side band transmission for transoceanic radiotele-
phone circuits and wire communications. In [1],[2], we see spectral analysis which
consists of estimating the instantaneous frequency of a signal.

In [5] using the DTA function instead of Daubechies scaling filter shifted by
$\pi/2$, gives a complex wavelet transform that satisfies a generalized linear phase,
orthogonality and reduction in shift sensitivity. Also another area which we believe
that our work can be extended, is in edge detection using local energy, [10]. The local
energy is calculated using the algorithm in [13]. Hence we believe that computing
the local energy using our method would be a good subject to investigate. Another
direction for future research is that instead of adding a constant to the DTA signal
obtained by [13], an alternative would be to add a linear sum of the original signal.
Then we would need to find a necessary condition for zeroing the DTFT of the new
DTA at several points in the negative frequency.
Bibliography


