A Wreath Product Group Approach to Signal and Image Processing:
Part II — Convolution, Correlation, and Applications

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Abstract

This paper continues the investigation of the use of spectral analysis on certain noncommutative finite
groups—wreath product groups—in digital signal processing. We describe here the generalization of discrete
cyclic convolution to convolution over these groups and show how it reduces to multiplication in the spectral
domain. Finite group-based convolution is defined in both the spatial and spectral domains and its properties
established. We pay particular attention to wreath product cyclic groups and further describe convolution
properties from a geometric viewpoint, in terms of operations with specific signals and filters. Group-based
correlation is defined in a natural way and its properties follow from those of convolution. We finally consider
an application of convolution: the detection of similarity of perceptually similar signals, and an application
of correlation: the detection of similarity of group transformed signals. Several examples using images are
included to demonstrate the ideas pictorially.

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1 Introduction

This paper continues the theoretical development of [6] and extends wreath product group-based signal processing to apply to two major properties: group-based convolution and correlation. We outline the generalization of discrete cyclic convolution to group-based convolution for an arbitrary finite group. This generalized convolution enjoys many of the properties of cyclic convolution, including transformation laws with respect to the group action. We consider group-based convolution and describe its properties. We also consider its use as a non-commutative filter in the detection of perceptually similar images. Group-based correlation is also defined and we consider its application in (peak) detection of group-transformed images.

Cyclic (circular) convolution $f \ast h$ of discrete signals $f$ and $h$ in a spatial (or time) domain plays a fundamental role in signal processing, particularly in the context of filtering. Each fixed $h$ gives a filter on the spatial domain by $f \mapsto f \ast h$. Different choices of $h$ extract different types of information from the inputs $f$. The classical DFT turns convolution in the spatial domain into multiplication in the spectral (frequency) domain. We shall see that for arbitrary finite groups the generalized Fourier transform also turns convolution in the spatial domain into matrix multiplication in the spectral domain. Moreover, when the representation of the group on the spatial domain is multiplicity-free—as is the case with the WPC groups—this matrix multiplication reduces to the multiplication of matrices by scalars. Thus by passing to the spectral domain, group-based convolution may be effectively computed. In particular, we saw in [6] that the analysis stage of an $M$-channel DFT filter bank outputs the spectrum of a discrete signal when the underlying group is a WPC group. Recalling that the WPC groups here are non-abelian, the associated convolution becomes non-commutative leading us to a whole new class of DFT-based non-commutative filters. This is one of the significant new additions of this work. Both convolution and correlation are illustrated with a number of examples to images.

1.1 Main Results

Although the theory is developed in full generality, for applications we confine ourselves primarily to the wreath product cyclic (WPC) group generated from the nine level tree with four branches descending from each node. Signals defined on the leaves of the tree are typically obtained here from a quadtree scan of $2^9 \times 2^9$ images. The main results are as follows:

(1) Group-based convolution for finite arbitrary groups and WPC groups is defined in both the spatial and spectral domains and its properties are established. We show how it may be effectively computed via the generalized Fourier transform and we explicitly consider the special case of WPC groups.

(2) A general formula for WPC convolution in terms of $\delta$-functions is established. WPC convolution is also interpreted in terms of some basic images and scale-selective filters.

(3) For images that are perceptually similar, we see that transforming them first through WPC group-based convolution generates a higher correlation coefficient than that obtained without transformation.

(4) Group-based correlation is defined naturally from group-based convolution and we see that it extends the peak matching property of correlation to signals (with a specific support) that are group transformed version of each other.
1.2 Organization and Notation

In order that the paper may be read smoothly without constant reference to [6], we recall the notation used earlier. Finite group-based convolution, its properties, and applications of WPC group-based convolution are described in Section 2. In a similar fashion, we address finite group-based correlation in Section 3. Section 4 summarizes our current findings, and we conclude with some directions for further work. In the Appendix we give a proof for the general formula for convolution for WPC groups acting on a quadtree.

Note that while we invoke two-dimensional images as examples to work with, and while this permits illustration of the WPC group spectrum in a standard two-dimensional “wavelet-type” decomposition, we still have a one-dimensional transform. Thus one-dimensional signals could also have been used for applications.

For convenience we summarize the notation from [6]; the numbers in parentheses indicate the section in [6] where each term is introduced.

\[ X = \{x_0, x_1, \ldots, x_{N-1}\} \] (3.1).

\[ G \] any finite group acting on X as permutations (3.1).

\[ L(X) = \{f : f : X \rightarrow \mathbb{C}\} \] the “spatial domain” of discrete signals of length N (3.1).

\[ L(G) = \{f : f : G \rightarrow \mathbb{C}\} \] the special case when X = G (also the complex group ring of G) (3.1).

\[ T_m \] the spherically homogeneous rooted tree, where \( m = (m_0, \ldots, m_{n-1}) \) (3.2.1).

\[ X_m \] the set of leaves of the tree \( T_m \) (3.2.1).

\[ Z_m \] the WPC group that acts on \( T_m \) (3.2.1).

\[ Q(k,n) \] the quadtree with n nonzero levels (3.2.2).

\[ Z(k,n) \] the WPC group that acts on \( Q(k,n) \) (3.2.2).

\[ L(X_m) = \{f : f : X_m \rightarrow \mathbb{C}\} \] the spatial domain indexed by the leaves of the tree (4.2).

\[ L(k,n) \] the spatial domain indexed by the leaves of the quadtree \( Q(k,n) \) (5).

\[ Q(f) \] the quadtree spectrum of f where \( f \in L(k,n) \) (5.1).

2 Finite Group-Based Convolution

We describe here finite group-based convolution and determine its properties. Since implementation in the spatial domain, which depends on the group order, can be computationally prohibitive in \( L(X_m) \) even for small trees, as 3.18 of [6] indicates, we consider its implementation in the spectral domain. Here convolution corresponds to a matrix multiplication of the spectra; for WPC groups in particular we see that it assumes an especially simple form in that it corresponds to multiplication of a matrix by a scalar.

Before considering group-based convolution further, we note other work in obtaining Fourier-like convolution theorems for multiresolution representations. While the spectral multiplication property does not hold for these representations, there has however been effort towards deriving linear convolution from the multiresolution spectra. A method for computing convolution with orthogonal and biorthogonal filter banks that reduces the determination of convolution to computing convolutions of the respective subbands and adding has been described by Vaidyanathan, [19]. A different subband convolution theorem, [20], for reducing convolution complexity, computes convolution using multirate filter banks and short-time Fourier transforms. Another technique for implementing convolution is given in [14]. None of these, however, provide the analogue to the convolution theorem wherein convolution in one domain becomes multiplication in the other. In that context, group-based convolution provides the exact analog. Group operations (corresponding to the shifting process in
cyclic convolution) result in a multiplicative property in the spatial domain, thereby entailing a product of the spectra.

For arbitrary finite groups a general formula for convolution with delta functions is established which allows computation of convolution for arbitrary signals \( f \) and \( h \). The general formula is made more explicit in the special case of WPC groups acting on the leaves of a quadtree \( Q(k, n) \) by defining convolution and its properties in terms of specific signals \( f \) and a class of scale-selective filters \( h \). This will permit easier visualization of the convolution process. We observe for the tree \( Q(1, n) \) that WPC convolution of signals \( f \) and \( h \), may be seen as the superposition of certain cyclic convolutions and averaging of subsequences of \( f \), weighted by elements of \( h \). A significant consequence of this is that WPC convolution generates sequences constant over successive lengths of size \( 4^k \), for \( k = 0, 1, \ldots, n - 1 \). From a filtering perspective, elements of \( h \) provide a scale-selective filtering operation on \( f \). This leads to a new class of linear, noncommutative filters that are both scale-selective as well as capable of convolving cyclically and smoothing.

We investigate the use of WPC convolution in determining signal similarity. Perceptually similar signals (images) \( f_i \) and \( f \) are compared using standard linear correlation, and then compared again after transformation by WPC convolution. For the examples tested, the WPC convolution transformed (lowpass) signals exhibit significantly higher correlation coefficients. Highpass signals, after specific WPC convolutions exhibit similar gains. This points to the possibility of using WPC convolution for problems in pattern recognition.

### 2.1 A Generalization of Cyclic Convolution

Let \( G \) be a finite group acting on a set \( X = \{x_0, x_1, \ldots, x_{N-1}\} \). Recall that for discrete signals \( f \) and \( h \) of length \( N \) (i.e., \( f, g \in L(X) \)), their cyclic convolution is

\[
(f * h)(x_n) = \sum_{m=0}^{N-1} f(x_m)h(x_{n-m}) = \sum_{m=0}^{N-1} f(\sigma^m x_0)h(\sigma^{-m} x_n),
\]

where \( \sigma \) is the \( N \)-cycle that cyclically permutes \( x_0, x_1, \ldots, x_{N-1} \), and the underlying group \( G \) is the cyclic group of order \( N \) generated by \( \sigma \). We define the formula for group-based convolution over an arbitrary finite group \( G \) acting on \( X \) which is the direct generalization. The group-based convolution of signals \( f \) and \( h \) in \( L(X) \), written henceforth as \( f * h \), is defined as

\[
(f * h)(x_n) = \frac{|X|}{|G|} \sum_{\beta \in G} f(\beta x_0)h(\beta^{-1} x_n), \quad \text{for all } x_n \in X.
\]

As with continuous and discrete cyclic convolutions, group-based convolution enjoys many properties (linearity, associative, etc.) which are most easily understood by passing to the spectral domain.

Following the development scheme in Section 3.1 of [6], we first describe convolution in \( L(G) \) and then describe how it restricts to the subspace \( L(X) \) of \( L(G) \). When \( G \) is a cyclic group acting faithfully and transitively on \( X \) we showed that \( L(X) \) may be identified with \( L(G) \), so the discussion of convolution on \( L(G) \) includes cyclic convolution as a special case.

The vector space \( L(G) \) has a natural multiplicative structure coming directly from the group operation on \( G \). For any \( \alpha \in G \), let \( f_\alpha \) denote the delta function in \( L(G) \) supported on \( \alpha \). Then

\[
f_\alpha \ast f_\gamma = f_{\alpha \gamma},
\]
where the product \( \alpha \gamma \) is taken in the group \( G \). This multiplication defined on the basis of delta functions extends uniquely by linearity and through the distributive law to a multiplication on all of \( L(G) \). In this way \( L(G) \) admits a multiplication that gives it the same structure as the group algebra of \( G \) over \( \mathbb{C} \). Moreover, it is easily seen that when \( X = G \), multiplication defined this way on \( L(G) \) is the same as convolution defined in (2.2)—the two formulas agree on the basis of delta functions since both products are bilinear (linear in each variable), where we choose an enumeration of the group elements such that \( x_0 \) is the identity element.

More generally, recall (cf. Section 3.1 of [6]) that when \( X \) is an arbitrary set acted upon transitively by \( G \), then \( L(X) \) is identified with the subspace \( L(G/G_0) \) of \( L(G) \) consisting of the functions on \( G \) that are constant on the left cosets of \( G_0 \), where \( G_0 \) is the subgroup of \( G \) fixing \( x_0 \). If \( f \) and \( h \) are constant on each left coset of any subgroup \( H \), then so is their convolution in \( L(G) \). Under the identification of \( X \) with the coset space \( G/G_0 \), the convolution formula (2.2) is the same as convolution multiplication in \( L(G) \) restricted to \( L(G/G_0) \).

The following properties are easily checked by the reader.

**Theorem 2.1** Let the finite group \( G \) act transitively on a set \( X \) and let \( f \ast g \) denote the group-based convolution of \( f \) and \( g \) in \( L(X) \) (which depends on both \( G \) and its action on \( X \)).

1. Convolution in \( L(X) \) is bilinear, and is associative.

2. Convolution is compatible with the (left) actions of \( G \) on \( L(G) \) and \( L(X) \): for the basis element \( f_\gamma \in L(G) \) and any \( \alpha \in G \)

   \[ \alpha f_\gamma = f_{\alpha \gamma} = f_\alpha \ast f_\gamma. \]  

3. The left action by \( \alpha \) on \( L(G) \) is linear, and so is left convolution by \( f_\alpha \). In particular, \( \alpha f = f_\alpha \ast f \) for all \( f \in L(G) \), and hence in \( L(G/G_0) = L(X) \) we have

   \[ \alpha f = f_\alpha \ast f, \quad \text{for all } f \in L(X) \text{ and all } \alpha \in G. \]  

4. Associativity, (2.4) and (2.5) together imply that convolution transforms under the group action by

   \[ \alpha(f \ast h) = (\alpha f) \ast h, \quad \text{and } f \ast (\alpha h) = (f \ast f_\alpha) \ast h \quad \text{for all } f, h \in L(X) \text{ and all } \alpha \in G. \]  

Note that when \( G \) is not commutative, convolution on \( L(G) \) is not commutative and convolution on its subspace \( L(G/G_0) \) may not be commutative.

Next we describe convolution viewed in the spectral domain; we first describe this in \( L(G) \), and then restrict to the subspace \( L(G/G_0) = L(X) \). By Wedderburn's Theorem (Theorem 4.1 of [6]), \( L(G) \) decomposes as a vector space into a direct sum of isotypic components, \( M_k \), of dimensions \( d_k^2 \). These components are also closed under convolution multiplication, and their structure is again described by Wedderburn's Theorem:

**Theorem 2.2** Let \( M_0, M_1, \ldots, M_{d-1} \) be the distinct isotypic components of \( L(G) \) described in Theorem 4.1 of [6]. Then \( M_k \) is the algebra, \( M_{d_k \times d_k}(\mathbb{C}) \), of all \( d_k \times d_k \) matrices with complex entries. Furthermore, \( L(G) \) is the direct product of these matrix algebras:

\[ L(G) \cong M_{d_0 \times d_0}(\mathbb{C}) \times M_{d_1 \times d_1}(\mathbb{C}) \times \cdots \times M_{d_{d-1} \times d_{d-1}}(\mathbb{C}), \]

where under this isomorphism the convolution of two functions in \( L(G) \) corresponds to the matrix product of their components on the righthand side (i.e., it is an algebra homomorphism as well).
This isomorphism implies that with respect to some basis of $L(G)$ chosen from the isotypic components $M_k$, the generalized Fourier transform of each $f \in L(G)$ may be written as a matrix in block diagonal form:

$$\hat{f} = \begin{pmatrix} \pi_0(f) & \pi_1(f) & \cdots & \pi_{r-1}(f) \\ \end{pmatrix},$$

where each $\pi_k(f)$ is now a $d_k \times d_k$ matrix representing the projection (4.4 of [6]) of $f$ onto the $k$th isotypic component $M_k$. Since this result holds for all discrete signals, in particular if $f$ and $h$ lie in the subspace $L(X)$ of $L(G)$ we have:

**Corollary 2.3** Convolution of functions in the spatial domain $L(X)$ corresponds to the (block diagonal) matrix multiplication of their spectra.

Thus the convolution in the spatial domain of two functions may be computed efficiently by passing to the spectral domain via a generalized Fourier transform, performing a matrix multiplication there, and passing back to the spatial domain by the inverse Fourier transform.

In the special case when the representation of $G$ on $L(X)$ is multiplicity-free, bases for the isotypic components may be chosen so that the block matrix description of the spectrum of each $f$ in $L(X)$ has a particularly simple form. (Recall that “multiplicity free” means that each isotypic component, $\pi_k(L(X))$, of the spatial domain is either 0 or irreducible.) These observations apply, in particular, to WPC groups acting on the leaves of a tree as in Section 3.2 of [6] (cf. Corollary 4.10 of [6]). In this case a basis of $L(G)$ may be chosen so that the elements of $L(X)$ are represented by block diagonal matrices whose blocks are zero in all but their first columns. We summarize this as follows:

**Lemma 2.4** If $G$ acts on $X$ in such a way that $L(X)$ is multiplicity-free, then there is a basis of $L(G)$ such that for every $k$ with $\pi_k(L(X)) \neq 0$ the matrices in (2.8) are of the form

$$\pi_k(f) = \begin{pmatrix} a_{11}^k(f) & 0 & \cdots & 0 \\ a_{21}^k(f) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{d_k^2}^k(f) & 0 & \cdots & 0 \\ \end{pmatrix}.$$  \hspace{1cm} (2.9)

With respect to this basis the Fourier transform of the convolution $f \ast h$ is the matrix product, $\hat{f} \hat{h}$, which is easily computed for block matrices described in (2.8) and (2.9): for each $k$ multiply every entry in the matrix block $\pi_k(f)$ by the upper lefthand entry of the corresponding block of $\hat{h}$, namely by $a_{11}^k(h)$.

In the case of multiplicity-free representations, each fixed $h \in L(X)$ acts as a linear filter on $L(X)$ by $f \mapsto f \ast h$. In the spectral domain right-multiplication by $\hat{h}$ then rescales each component of $\hat{f}$ by a constant (which depends on the component). Different signals $h$ may act as the same filter, since for $i \geq 2$ the entries $a_{11}^k(h)$ of $\hat{h}$ in (2.9) have no effect on the product $\hat{f} \hat{h}$.

Some specific choices of filters $h$ are of special interest.
Example 1. Let $\delta_{x_0}$ be the unit impulse delta function in $L(X)$ supported on $x_0$. Then $\delta_{x_0}$ is the delta function in $L(G)$ supported at the identity element of $G$ projected onto the subspace $L(X)$. Thus $f * \delta_{x_0} = f$ for all $f \in L(X)$ i.e., $\delta_{x_0}$ is a right identity of $L(X)$. The Fourier transform of $\delta_{x_0}$ has a one in the upper lefthand entry of each block $\pi_k(\delta_{x_0})$ and zeros in all other entries.

Example 2. Let $h_k \in L(X)$ be the signal whose Fourier transform $\hat{h}_k$ is the delta function in the spectral domain with a 1 in the upper left entry of block $\pi_k(h_k)$ and zeros in all other entries of all blocks. In other words $h_k$ is a bandpass filter: the Fourier transform of $f * h_k$ is zero in all but the $k^{th}$ component, and $f * h_k$ has the same $k^{th}$ component as $f$. Thus the filter $h_k$ is the $k^{th}$ component of $\delta_{x_0}$, and so may be computed by projecting $\delta_{x_0}$ onto this component, as described by Theorem 4.2 of [6]:

$$h_k(x_j) = \pi_k(\delta_{x_0}) = \frac{d_k}{|G|} \sum_{\beta \in G_k} \chi_k(\alpha_j, \beta), \quad \text{for all } x_j \in X,$$

(2.10)

where $\alpha_j$ is any element of $G$ that sends $x_0$ into $x_j$, and $d_k$ is the dimension of the $k^{th}$ isotropic component of $L(X)$.

2.2 Convolution Over WPC Groups

In this section let $G$ be the WPC group $Z_m$ acting on the leaves of the tree $X_m$. In this case the convolution $f * h$ defined by (2.2) is called WPC convolution. Since WPC convolution may require as many as $|G|$ multiplications to compute each value, as noted earlier this method can be computationally prohibitive even for small trees. It is computationally more effective to pass to the spectral domain; this is especially efficient since the representation of $Z_m$ on $L(X_m)$ is multiplicity-free. The bases in Section 4.2 of [6] giving the decomposition of $L(X_m)$ into irreducible components are also precisely the ones that give the isomorphism of multiplicative structures described in (2.7), (2.8), and (2.9). In particular, the generalized Fourier coefficients are precisely the matrix entries in (2.9).

We may easily describe convolution in the case when $G = Z(k,n)$ and signals $f$ and $h$ in $L(k,n)$ have a quadtree spectrum (cf. Section 5.1 in [6]). Both spectraums $Q(f)$ and $Q(h)$ are formed from submatrices containing the coefficients of the representations of $Z(k,n)$ on the irreducible subspaces, with bases compatible with convolution computations. The upper lefthand entry of each of these submatrices is the coefficient of the corresponding bandpass signal, $h_k$. Consequently, we have the following result.

Theorem 2.5 Convolution of signals $f$ and $h$ in $L(k,n)$ may be computed in the spectral domain by multiplying each entry in each nested grid irreducible submatrix of $Q(f)$ by the upper lefthand entry of the corresponding block matrix of $Q(h)$.

Example. Following the notation of Example 1 in Section 5.1 in [6], if $f$ and $h$ are in $L(1,2)$, denote $Q(f)$ and $Q(h)$ by the $4 \times 4$ matrices $(a_{i,j})$ and $(b_{i,j})$ respectively. Then

$$Q(f) \cdot Q(h) = \begin{pmatrix}
a_{1,1}b_{1,1} & a_{1,2}b_{1,2} & a_{1,3}b_{1,3} & a_{1,4}b_{1,4} \\
a_{2,1}b_{2,1} & a_{2,2}b_{2,2} & a_{2,3}b_{2,3} & a_{2,4}b_{2,4} \\
a_{3,1}b_{3,1} & a_{3,2}b_{3,2} & a_{3,3}b_{3,3} & a_{3,4}b_{3,4} \\
a_{4,1}b_{4,1} & a_{4,2}b_{4,2} & a_{4,3}b_{4,3} & a_{4,4}b_{4,4}
\end{pmatrix}.$$
We now describe the spectral multiplication for the quadtree spectra of $2^{kn} \times 2^{kn}$ images $f$ and $h$ in $L(k,n)$ (although, as is our style, we only detail the process for $k = 1$). WPC convolution of signals in general is likewise easily effected (cf. Theorem 2.2). If $Q(f) = (a_{pq})$ and $Q(h) = (b_{pq})$ are the $2^n \times 2^n$ quadtree spectral matrices of $2^n \times 2^n$ images $f$ and $h$ respectively, their product in the spectral domain is computed as follows: working in stages from $j = 0$ to $n$, divide the upper left hand $2^{n-j} \times 2^{n-j}$ submatrix of $Q(f)$ into a grid of four blocks:

$$
\begin{pmatrix}
U_0 & U_1 \\
U_3 & U_2
\end{pmatrix},
$$

where each $U_p$ is a $2^{n-j-1}$ square submatrix. At the $j$th stage multiply every entry of the block $U_p$ by the upper left hand entry of the submatrix of $Q(h)$ in the same location as $U_p$, for $p = 1, 2, 3$. In other words:

1. step 1: multiply each entry of $U_1$ by $b_{1,2^{n-j-1}+1,1}$,

2. step 2: multiply each entry of $U_2$ by $b_{2^{n-j-1}+1,2^{n-j-1}+1}$,

3. step 3: multiply each entry of $U_3$ by $b_{2^{n-j-1}+1,1}$.

The submatrix $U_0$ is passed unaltered to the next stage where $j \rightarrow j + 1$ and the process is repeated in the smaller matrix $U_0$. At the last ($j = n$) stage multiply $a_{1,1}$ by $b_{1,1}$.

This shows that WPC convolution is not generally commutative, nor is it unique in the right variable $h$. Different filters $h$ giving the same product $f \ast h$ are obtained by simply altering the spectrum of $h$ in any position except the upper left hand entries of the submatrix blocks. Convolution is invertible when all these entries of the spectrum of $h$ are nonzero.

### 2.3 WPC Convolution via Scale-Selective Filters

In this section we consider WPC convolution from the geometric point of view. In particular, we analyze in detail the effect of convolution on images, which are interpreted as functions on the leaves of regularly branching trees (cf. Section 3.2.2 of [6]). For simplicity we work with trees of the form $Q(1,n)$; corresponding results for $Q(k,n)$, $k > 1$, follow easily.

#### Theorem 2.6

Let \{x_0, x_1, \ldots, x_p, \ldots, x_{4^n-1}\} be the set of leaves of the tree $Q(1,n)$ acted upon by the WPC group $G = Z(1,n)$. Let $G_0$ be the subgroup of $G$ fixing leaf $x_0$. For any index $i$ let $h_i \in L(1,n)$ be the unit impulse delta function supported at $x_i$, $0 \leq i \leq 4^n - 1$. Denote $h_0$ by $f_0$. Then the following hold:

1. For any index $i > 0$ the convolution $f_0 \ast h_i$ is the function defined by

   $$
   (f_0 \ast h_i)(x_p) = \begin{cases} 
   0 & \text{if } x_i \text{ and } x_p \text{ do not lie in a common subtree not containing } x_0 \\
   4^n & \text{if the largest subtree containing } x_i \text{ and } x_p \text{ but not } x_0 \text{ has root on level } s,
   \end{cases}
   $$

   where $f_0 \ast h_0 = f_0$.

2. If for each $j$ we let $a_j$ be an element of $G$ sending $x_0$ to $x_j$, then

   $$
   (f \ast h_0)(x_p) = \sum_{j=0}^{4^n-1} f(x_j) a_j (f_0 \ast h_0)(x_p) = \sum_{j=0}^{4^n-1} f(x_j) (f_0 \ast h_i(a_j^x x_p)).
   $$

3. The linearity of convolution in the second variable reduces the computation of $f \ast h$ for an arbitrary $h$ to linear combinations of convolutions of the types in parts (1) and (2).
Proof: See Appendix A.

For $2^d \times 2^d$ images scanned in the $Q(1,9)$ quadtree fashion the convolution formula in part (1) of the theorem may be visualized as follows. The signal $f_0$ is represented by the image with a 1 in position 1,1 and zeros elsewhere. Let $h_{i,j}$ be the image with 1 in position $i,j$ and zeros elsewhere. Let $Q_d^e$ be the unique largest $2^{d-s} \times 2^{d-s}$ square among the nested grid subquadrants depicted in Figure 8 of [6] that contains the $i,j$ pixel but not the 1,1 pixel, where we choose $Q_d^e = Q_d^1$ when $i,j = 1,1$. Theorem 2.6 asserts that the convolution image $f_0 \ast h_{i,j}$ is the array with value $4^{s-9}$ in each pixel of the square $Q_d^e$ and zeros in all other positions. (Note that the quadrants $Q_d^r$ with $r = 1,2,\ldots,8$ and $d = 1,2,3$ together with the 1,1 entry $Q_d^1$ are the complete set of orbits of the stabilizer subgroup $G_0$ acting on the quadtree array, and $G_0$ acts independently as the full WPC group $Z(1,9 - r)$ on each $Q_d^r$; this gives an explicit visualization of the decomposition of $G_0$ in the displayed line (A2) of the proof.

2.3.1 A Geometric View

Because cyclic subgroups comprise the wreath product factors of the WPC group $Z(k,n)$, cyclic convolution is inherent in WPC convolution. However, overall effects are quite different from those obtained with either cyclic or standard convolution. As observed in Theorem 2.6, a WPC group induces effects in the convolution process that are dependent on the location of the support of both $f$ and $h$ on the leaves of the tree $Q(k,n)$. This induces a scale-based filtering of $f$. Accordingly, to understand the process of convolving signal $f$ with filter $h$, we again define signals and filters with specific support.

Define $f_C$ to be any signal with support concentrated on the first $4^k$ leaves of the tree $Q(k,n)$ (i.e., those descending from the $0^{th}$ node at level $n-1$). In our quadtree correspondence, interpreting $f_C$ as an image, it is nonzero only in the upper left $2^k \times 2^k$ block $Q_0^{k-1}$. (Recall that the superscript accompanying $Q$ signifies the level.) By Theorem 2.1, convolution of any arbitrary $f$ in $Q(k,n)$ is determined by the knowledge of convolution of $f_C$ with the filters $h_i$. We illustrate with the special case $k = 1$ and $n = 9$, and refer to Sections 3.3 and 5.2 of [6]. As indicated there, through the quadtree scan the $i^{th}$ level of the WPC spectrum is seen to lie in subquadrants 1, 2 and 3 (where quadrant 0 is the upper left one). Since signal $f_C$ defined on the $2 \times 2$ quadrant $Q_0^0$ is subject to averaging operations in the convolution process, we define the following: let $s$ be the sum of the values of $f_C$, and for each $i = 0,1,2,\ldots,8$ define $f_{C_i}$ by

$$f_{C_i} = \text{the image with value } s/4^{k-2} \text{ at each pixel in quadrant } Q_0^i, \text{ and 0 elsewhere,}$$

(2.13)

as depicted in Figure 1. Thus $f_{C_i}$ is constant on the upper left quadrant at level $i$, is zero outside this quadrant, and has the same DC term (sum of all its values) as $f_C$.

For filter $h$ we consider the following set of filters for the tree $Q(1,9)$. Changing the earlier labeling slightly to accommodate two-dimensional images, we now let

$$h_0 = \begin{cases} 1 & \text{in position } 1,1 \\ 0 & \text{elsewhere} \end{cases} \quad h_i = \begin{cases} 1 & \text{in position } 1,1 + 2^{i-1} \\ 0 & \text{elsewhere} \end{cases} \quad 1 \leq i \leq 9,$$

(2.14)

as depicted in Figure 2. For $1 < i < 9$ define an extension of filter $h_i$ to be an image with a unit impulse located anywhere in quadrant $Q_1^{10-i}$. We abuse this definition slightly and call all the unit impulse delta functions
supported in the upper left-hand $2 \times 2$ block extensions of filters $h_0$ or $h_1$ (so $h_1$ and $h_0$ are considered extensions of each other). The generators for the group $Z(1,9)$ described in Section 3.3 of [6] permute the filters $h_i$ in the sense that $h_i = a^{(0-i)}_0(h_0)$, $1 \leq i \leq 9$, where $a_0$ denotes the 0th generator from among these families acting at level $i$, permuting the four trees descending from node 0 at level $i$. Moreover, all generators acting at levels below $i$ simply permute the extensions of $h_i$ among themselves (with many such group elements actually fixing $h_i$).

By Theorem 2.1, for filter $h_0$ we have

$$f_C \ast h_0 = f_C \quad \equiv g_1$$

$$f_C \ast (a_0^{(8)})^k(h_0) = (a_0^{(8)})^k(f_C) = (a_0^{(8)})^k(g_1) \quad k = 0, 1, 2, 3,$$

where $(a_0^{(8)})^k \in Z(1,9)$ is the group operation at level 8, acting on the first four leaves. Note that extensions of $h_0$, that is, filters supported on $Q_8^8$ (the first 4 leaves of the tree), convolved with signal $f_C$ (also defined on $Q_8^8$) results in the standard cyclic convolution on the set $Q_8^8$. Hence in two dimensions, Theorem 2.1 implies that an image $f$ convolved with filter $h_0$ and its extensions results in an in-place circular convolution of all $2 \times 2$ blocks of the image with the filter. We proceed in this fashion, examining the effect of group operations on filter $h_i$ only at level $9-i$ for $i = 2, 3, \ldots, 9$. Group operations at higher levels are seen to have no effect, while those at lower levels are formulated in terms of the corresponding $h_i$.

For filter $h_2$, by again using Theorem 2.1 we have

$$f_C \ast h_2 = f_C \ast (a_0^{(7)})(h_0) = \equiv g_2$$

$$f_C \ast (a_0^{(8)})^k(h_2) = f_C \ast (a_0^{(7)})(a_0^{(8)})^k(h_0) = (a_0^{(7)})(f_C) \quad k = 0, 1, 2, 3,$$

Thus $f_C$ convolved with $h_2$ and its extensions generates output $g_2$ which is $f_C$, but with support on subquadrant $Q_2^8$. Equivalently, $g_2$ is $f_C$ operated on by group element $a_0^{(7)}$. Local group operations $(a_0^{(8)})^k$ on $h_2$ do not affect the output $g_2$. However, group operations $(a_0^{(7)})^k$ on $h_2$ and its extensions that move the impulse in quadrant $Q_2^8$ result in corresponding permutations on $g_2$.

Extending our analysis this way, we see that convolving signal $f_C$ with the other unit impulse filters $h_i$ for $i = 3, 4, \ldots, 9$ and their extensions results in an output $g_i$ consisting of averages $f_{C_{10-i}}$ located in quadrants $Q_1^{10-i}$, the same as that for filters $h_i$. Accordingly, for filter $h_0$ we have

$$f_C \ast h_0 = f_C \ast (a_0^{(0)})(h_0) = (a_0^{(0)})(f_C) \quad \equiv g_0$$

$$f_C \ast (a_0^{(0)})^k(h_0) = (a_0^{(0)})^k(g_0) \quad k = 0, 1, 2, 3,$$

Finally, by the bilinearity of WPC convolution and Theorem 2.1, we see that $f \ast h$ consists of a superposition of cyclic convolutions and averaging operations. All 4-element sequences of $f$ are cyclically convolved in-place with the 4-element sequence of $h$ defined on quadrant $Q_0^8$. A filter $h$ supported on quadrants $Q_0^8$ and convolved with $f_C$ generates weighted averages of $f_C$ on the same quadrants $Q_0^k$, for $k = 8, 7, \ldots, 1$ and $j = 1, 2, 3$. The remaining elements of $f$—which are of the form $\tau(f_C)$ for some $\tau \in G$—when convolved with $h$ appear as rotated (by $\tau$) versions of the operations on $f_C$. Hence, we observe that the convolved output $g = f \ast h$ appears as a pattern in blocks of size $4^k$, for $k = 0, 1, \ldots, 8$; the cyclic convolution effects appear in all 4-element sequences.
(k = 0), while the weighted averages appear in sequences of increasing size \(4^k\), for \(k = 1, 2, \ldots, 8\). In summary, we conclude:

**Theorem 2.7**

1. Filter \(h_0\) and its extensions cyclically convolve in-place blocks of \(f\) of size \(2 \times 2\).

2. For \(i = 2, 3, \ldots, 9\), filters \(h_i\) and their extensions act as scale-selective filters, and they average \(f_C\) over a block of size \(2^{i-1} \times 2^{i-1}\). The averaging results \(g_i\) are supported in subquadrants \(Q_0^{9-i}\), the same as the filter; thereby reflecting a local rotation of the image. Group operations on \(h_i\) acting at level \(9 - i\) result in the corresponding group operation on the outputs \(g_i = f_C * h_i\), for \(i = 2, 3, \ldots, 9\).

3. WPC convolution \(f \ast h\) of arbitrary images is then obtained using the transformation of convolution under group operations on \(f_C\) and \(h_i\) and bilinearity.

A correcting operation whereby scale-selective filtering of \(f\) appears coincident with \(f\) can be obtained by simply convolving \(g_i\) as defined above, with appropriately rotated versions of \(h_i\). That is, \(g_i \ast (\tau_i^2(h_i))\) where \(\tau_i = \alpha_0^{9-i}\) for \(i = 2, 3, \ldots, 9\) respectively, generates output averages \(f_{C_{0-i}}\). Accordingly, for image \(f_C\), scale-filtered images \(f_{C_{LP_i}}\) and the corresponding scale-selective lowpass (LP) filters \(h_{LP_i}\) are obtained as

\[
\begin{align*}
    f_{C_{LP_i}} &= g_{i+1} \ast (\tau_i^2(h_{i+1})), \\
    &= f_C \ast h_{i+1} \ast (\tau_i^2(h_{i+1})) \\
    &= f_C \ast h_{LP_i},
\end{align*}
\]

where, by the associative property of convolution, \(h_{LP_i} = h_{i+1} \ast (\tau_i^2(h_{i+1}))\) is the average of \(h_{i+1}\) uniformly distributed on the first \(4^i\) leaves of the tree, or equivalently, with support on quadrant \(Q_0^{9-i}\). Hence, given an image \(f\), its lowpass scale-selective filtered image \(f_{LP}\), defined as \(f \ast h_{LP}\), may be obtained using the group transformation and bilinearity properties of convolution. Defining \(h_{LP_0}\) as the average of \(h_0\) uniformly distributed on quadrant \(Q_0^9\) (the whole image), it is easily seen that convolving \(f\) with \(h_{LP_0}\) generates the average value of \(f\) over quadrant \(Q_0^9\). Figure 3 shows the structure of the lowpass filters and their associated spectra. Hence \(f\) convolved with, for example, \(h_{LP_9}\) generates the average of \(f\) in quadrants \(Q_1^9\), for \(i = 0, 1, 2, 3\), placing the average in the same quadrants. Recall that multiresolution subspaces \(V_i\) and \(W_i\) of \(L(1, 9)\) were defined in Section 5.2 of [9] as being associated to level \(i\), for \(i = 0, 1, \ldots, 9\), where \(V_0\) corresponded to the highest detail and \(V_9\) to the coarsest. We now associate \(V_i\) and \(W_i\) as being at scales \(9 - i\), for \(i = 0, 1, \ldots, 9\), where small scale corresponds to high resolution. Hence, identifying filters \(h_{LP_i}\) as scale-selective lowpass filters, we see that convolving \(f\) with \(h_{LP_i}\) generates scale-selective reconstructions of \(f\) at the nine scales or levels. For \(i = 1, 2, \ldots, 9\), \(h_{LP_i}\) filters spectra in subspaces \(V_0-i\), and rejects spectra in subspaces \(W_0-i\) and all lower scales. In terms of filtering we thus have:

**Theorem 2.8** WPC convolving \(f\) with \(h_{LP_i}\) results in a lowpass zonal filtering of the spectrum, successively filtering out details at scales \(i\), for \(i = 1, 2, \ldots, 9\) and those below it.

While \(h_i\), for \(i = 2, 3, \ldots, 9\), and its extensions act as scale-selective lowpass filters, we have seen that \(h_0\) and its \(2 \times 2\) extensions act as local cyclic convolvers with \(2 \times 2\) blocks of the image \(f\). However, cyclic convolution can also generate scale-selective filters through an appropriate choice of filter coefficients. For example, consider
a scaled extension of \( h_0 \) defined by \( h_{HR_1} = [h_1 \ h_2 \ h_3 \ h_4] = [\sqrt{2} \ -0.25 \ -0.25 \ -0.25] \). It is easily seen that its spectrum, \( H_{HR_1} \), consists of ones located in the 1,1 term of the quadtree spectral subspaces \( W_{8,1}, \ W_{8,2}, \ W_{8,3} \) and zeros elsewhere. Hence convolving \( f \) with \( h_{HR_1} \) results in a scale 1 highpass filtering of \( f \). Thus the spectrum of \( f \) is altered such that coefficients at all scales except the highest resolution subspaces \( W_{8,1}, \ W_{8,2}, \ W_{8,3} \) are set to zero. This is equivalent to highpass filtering at scale 1, giving the detail at scale 1.

In a similar fashion we can generate other scale-selective filters \( h_{HR_i} \), for \( i = 2, 3, \ldots, 9 \), that act as bandpass filters, providing the detail at those levels. That is, these filters will filter only the spectrum at scales 2 through 9, corresponding respectively to subspaces \( W_{i,j} \), for \( i = 7, 6, \ldots, 1, 0 \) and \( j = 1, 2, 3 \). Such bandpass filters may be obtained by extending the support of \( h_{HR_1} \) from a \( 2 \times 2 \) grid to a \( 2^i \times 2^i \) grid, wherein each element of \( h_{HR_1} \) is averaged over a \( 2^{i-1} \times 2^{i-1} \) region to form \( h_{HR_i} \), for \( i = 2, 3, \ldots, 9 \). The spectrum, \( H_{HR_i} \), of \( h_{HR_i} \) is nonzero in subbands such that filtering \( f \) with \( h_{HR_i} \), for \( i = 1, 2, \ldots, 9 \), filters subbands \( W_{9-i} \) to give scale 1 to 9 reconstruction of the detail images of \( f \). Figure 4 shows the construction of the bandpass filters. Thus we have:

**Theorem 2.9** Filters \( h_{LP_i} \) and \( h_{HP_i} \) form scale-selective lowpass and bandpass filters. Since WPC convolution is bilinear, a variety of scale-selective filters may be constructed through a combination of these lowpass and bandpass filters.

While an averaging interpretation has been provided for filters \( h_i \), for \( i = 2, 3, \ldots, 9 \), we can also interpret their action in terms of cyclic convolution, as we did for \( h_0 \). Since convolving \( f_C \) with \( h_2 \) generates an average \( f_{C_2} \), this is easily seen to be equivalent to convolving \( f_C \) with \( h_2 \), where \( h_2 \) is the average of \( h_2 \) over \( Q_2 \). Now \( f_C \ast h_2 \) gives the cyclic convolution of the two 4-point sequences, which is \( f_{C_2} \) (in \( Q_2 \)). Similar interpretations follow for convolving \( f_C \) with \( h_i \) for \( i = 3, 4, \ldots, 9 \), and with their extensions. Convolution of a \( 256 \times 256 \) image \( f \) with three filters \( h_{LP_1}, h_{LP_2}, \) and \( h_{LP_3} \) is shown in Figure 5.

From the convolution properties described we reinforce the previously observed fact that different filters \( h \) may produce the same outputs \( f \ast h \) for a given signal \( f \). An arbitrary filter \( h \) may, for example, be replaced by a \( h_{EQ} \) where \( h_{EQ} \) is constructed from \( h \) by replacing every element of \( h \) in quadrants \( Q_{ij} \) by its average value in the respective quadrants of \( h_{EQ} \), for \( i = 1, 2, \ldots, 8 \) and \( j = 1, 2, 3 \). Equivalently, \( h \) may be replaced by the sum of \( h \) in, say, the 1,1 terms of the respective quadrants, or by other descriptions as long as the corresponding sums in \( Q_{ij} \) in both \( h \) and \( h_{EQ} \) are preserved. Regardless of the equivalent \( h \) utilized, the process of convolving \( f \) with a given filter \( h \) (on the right) is always achieved by convolving \( f \) with the same canonical filter \( h_{CN} \) whose spectrum is multiplied by that of \( f \). The canonical filter is defined as that whose spectrum in all the irreducible subspaces (i.e., subquadrants) is identical to the 1,1 term in the corresponding subspaces (i.e., in the upper left-hand entry). Hence convolving \( f \) with \( h \) always amounts to convolving \( f \) with \( h_{CN} \). For a convolution of a \( 256 \times 256 \) image \( f \) with itself, i.e., with the filter \( h = f \), we see in Figures 6(a), (b), (c) and (d): the image \( f \) and filter \( h \), an equivalent filter \( h_{EQ} \), the canonical filter \( h_{CN} \), and the resulting convolved image \( g \). With \( H \) representing the spectral matrix of \( h \), \( h_{EQ} \) here corresponds to a filter whose spectral matrix \( H_{EQ} \) retains only the 1,1 terms of \( H \) in the four subquadrants of \( Q_0 \) and \( Q_2 \), for \( i = 6, 5, \ldots, 1 \) and \( j = 1, 2, 3 \), and sets all other terms to zero. Note that this \( h_{EQ} \) is also \( h_0 \ast h_0 \), which we define as the impulse response of filter \( h \). Accordingly, the non-uniqueness of \( h \) is illustrated by \( h_{EQ} \) and \( h_{CN} \). We also note that for the tree \( Q(k, n) \), only when the image has support on the first \( 4^k \) locations of the tree does the impulse response become identical to the filter.
2.4 WPC Convolution and Scale Similarity

Having formulated the notion of convolution for WPC groups we now investigate some potential filtering applications. In particular, we apply it to the similarity determination of two signals. Our method is to first transform a signal through WPC convolution and then measure distance (“closeness”) in that space. Two examples, one synthetically generated and the other consisting of natural images, are shown to illustrate the application.

2.4.1 Distance, Similarity, and Correlation

The notion of distance of two signals takes many forms in the literature. In signal processing, minimizing various metrics leads to the design of optimal filters. For example, the $L^\infty$-norm leads to the Chebyshev filter [5], while maximally flat approximations in the Taylor series sense generate the classic Butterworth and Bessel filters [9] (pp.245–250), and [13]. For signal-noise problems, using the $L^2$-norm leads to the matched and Wiener filters [17]. In pattern recognition, distance is often measured by the nearest neighbor classifier which assigns a test signal to a class whose template vector is closest or most “similar” to it. While many different functions have been described in the literature to measure similarity, as cited by Hadar et al., [8], correlation (i.e., the standard $l_2$-correlation) or template matching has found wide use in the image processing literature. Its use in measuring similarity dates back to the very earliest efforts [4] in pattern recognition. Previous work of Anuta [2] and Ballard [3] describes how the correlation coefficient serves as a good measure for a variety of tasks including registration and sterosis of images. Use of the correlation coefficient tends to normalize out intensity differences that might adversely affect the match in a spurious way. Many other applications of correlation in image analysis are described in [8]. Calculating correlation through multiresolution representations has recently been investigated by Stone [15] using the earlier work of Vaidyanathan [19]. Correlations coefficients are calculated by correlation operations that are computed using Fourier and wavelet-packet transforms. Application to hierarchical image analysis, where the goal is to correlate the template and test signal at various scales, is given in [15]. In that approach matching is performed sequentially from coarse to fine scale, proceeding from one level to the next after the corresponding cost function becomes minimal.

The specific application we consider here falls generally within the above framework. We have seen that the correlation coefficient serves as a good indicator for measuring similarity when the signals are not necessarily identical but are linearly correlated. Normalization of correlation makes the metric less dependent on local properties, such as image intensity differences, that can adversely affect the ability to match images. However, such an adjustment does not allow for important factors such as images at different scales, or other factors such as imaging distortions, rotations, and texture differences. In such cases we can have perceptually similar, that is qualitatively similar images but yet have low correlation values that may imply a mismatch and thus vitiate the advantages of normalized correlation. To address the problem of low correlation for images at different scales we investigate the effects of measuring the correlation of images transformed to a WPC representation, where perceptually similar images can become measurably closer. The technique of applying local linear transformations has been adopted by Picard [12] and others [18], [1], all utilizing some application of the Karhunen-Loève transform. Picard, for example, in a texture matching problem, uses subsets of features obtained from the Karhunen-Loève transform, and compares these through a mean-square error criterion. The transform itself is generated from the covariance matrix of a small random subset of DFT magnitudes of the data set. In the approach adopted here, we attempt to match images, especially those generated at various scales, by first
transforming them to a multiresolution representation and then using normalized correlation. For determining
similarity of a scaled image $f_i$ to a prototype image $f$ we compare $f_i * f$ and $f * f$ using standard correlation
and compare that to the correlation of $f_i$ and $f$. We carry out this analysis by utilizing the WPC multiresolu-
tion representation: for the tree $Q(1, 9)$, using the decomposition $f = \sum_{j=1}^{8} f^j$, where $f^j$ is $f$ defined over
quadrants $Q_0^8$ and $Q_1^1 + Q_2^1 + Q_3^1$ for $i = 8, 7, \ldots, 1$, we have seen that WPC convolution of signal $g$ with $f$ is the
superposition of signals constant over successive lengths of size $4^k$, for $k = 0, 1, \ldots, 8$. Specifically, $f_i * f$ gives a
weighted decomposition of $f_i$ in all 9 subspaces, consisting of a cyclic convolution and the sum of weighted by
$f^j$ averages of $f_i$ that are projections on subspaces $W_8, W_7, \ldots, W_0$ and $V_0$ (discussed prior to Theorem 2.8).
We use this representation for comparison.

**Example 1.** As a first example we see in Figures 7(a) and (b) two $256 \times 256$ test images $f_1$ and $f_2$ obtained
from the WPC convolution of the prototype image $f$ of Figure 6(a) with filters $h_{LP}$ and $h_{HP}$. Accordingly, $f_1$
and $f_2$ being respectively, projections of $f$ at scale 1 lowpass and of $f$ at scale 2 highpass, have spectra that lie
in certain fixed bands. We attempt to measure similarity of both to $f$. Note that for the synthetic image case
here, a simple difference of the WPC spectra of $f_1$ and $f$ would yield the fact that both have a scale similarity
to $f_1$. However, for the general case where similarity may not be exact or where the two images may stem from
different multiresolution bases, we try the more general approach of WPC convolution indicated above.

Consider first the lowpass image $f_1$. By construction $f_1$ is dissimilar to $f$ at scale 1 highpass where it is
zero, but similar at all other scales. Accordingly, the spectrum of $f_1 * f$ is similar to that of $f * f$ at all scales
except scale 1 highpass, where the former is zero. Consequently, all successive four-point sequences in $f_1 * f$ will
be constant. Since their sum is kept the same as in the corresponding part of signal $f * f$, the spectrum of the
two convolutions is the same at all other scales. The resulting effect is that the two convolution patterns will
be similar at all scales except scale 1 highpass, where $f_1 * f$ will be constant. We observe the general similarity
in Figure 7(c). The dissimilarity at scale 1 is seen by observing the 4-point sequences in Figure 7(e). Here
we observe that $f_1 * f$ is constant in all successive 4-point sequences while $f * f$ is not. Similarity at scales
greater than 1 is manifested by the fact that $f_1 * f$ has the same frequency characteristics as $f * f$ for sequences
considered in successive blocks of length $4^k$, for $k = 1, 2, \ldots, 7$.

We devise a bandpass image $f_2$ similar to $f$ at scale 2 highpass but dissimilar at others, where it has zero
spectrum. Spectra of the resulting convolutions $f_2 * f$ and $f * f$ carry those same characteristics. Figure 7(d)
shows the two convolutions, which clearly appear dissimilar. Examining convolutions at the scale of interest,
namely scale 2, we see two typical 16-point sequences in Figure 7(f). We observe that both convolutions have
the same frequency characteristics at that scale, that is, for successive segments of length $4^2$, taken in blocks of
size 4. Dissimilarity at scale 1 is discernible. Dissimilarity at other scales exists due to $f_2 * f$ having a zero sum
over successive sequences of length $4^k$, for $k = 2, 3, \ldots, 7$.

We now correlate to measure signal similarity. Before transformation the correlation coefficient, $\rho_i$ of $f_i$ and
$f$ is 0.9421 and 0.3817 for $i = 1, 2$ respectively. After WPC convolution, as described above, it is 1 and 0.0019.
A corresponding effect holds for $f_i$ similar to $f$ at the other lowpass and highpass scales. Hence for this example
and for the lowpass case, transformation using WPC convolution provides greater normalized correlation than
that obtained without convolution.

A correlation coefficient of 0.3817 is not high, and would typically not imply a good match. For the WPC
case however, since the low value of 0.0019 is due to matching at only 1 (low-energy) scale and a mismatch at
all others, it would appear logical to compare convolutions at each scale rather than simultaneously across all scales. Accordingly, we compare $f_2 * h_{HP_i}$ and $f * h_{HP_i}$ for $i = 1, 2, \ldots, 8$, where $h_{HP_i}$ are the highpass filters of Figure 4. Using covariances, since the sum of the values of the projections of $f_2$ are zero for $i = 1, 2, \ldots, 8$, we get, as expected, covariance of value 0 at all scales except scale 2 where $\rho$ is 1. Hence we are able to establish high correlation for this highpass signal using projections at each scale. Accordingly, convolving to obtain a multiresolution representation for the lowpass case, and at each scale as in the highpass example, results in a characterization that generates higher correlation values than those obtained directly.

2.4.2 Relationship between correlation coefficients

Before proceeding with a more general example we first establish relationships between the $l_2$-norms of signals before and after WPC convolution. We consider again the tree $Q(1,9)$. Recall that for $f, g \in L(1,9)$, the convolution $g * f$ achieved via spectral multiplication uses WPC spectra without normalization. Since only 4-point DFTs are involved, $l_2$-norms are preserved if scaling factors are introduced. Hence it is easily seen that for a signal $g$ and spectrum $\hat{g}$ we have

$$
||g||^2 = \sum_{n=1}^{d} |g(n)|^2 = \sum_{j=1}^{3} \sum_{n_{i,j}} |\hat{g}_{i,j}|^2 + \cdot \cdot \cdot + \sum_{k=1}^{3} \sum_{n_{i,j}} |\hat{g}_{i,j}|^2 + \sum_{k=0}^{1} |\hat{g}_{0}|^2,
$$

(2.15)

where $\hat{g}_{X}$ refers to the normalized spectrum of $g$ in the irreducible subspaces $W_{i,j}$ of Section 5.2 of [6]. For WPC convolution $g * f$ we then have

$$
||g * f||^2 = ||\hat{g}h_{s,1}||^2 |f_{s,1}|^2 + \cdot \cdot \cdot + ||\hat{g}h_{s,9}||^2 |f_{s,9}|^2,
$$

(2.16)

where $f_{k,1}^j$ denotes the 1,1 term of the spectrum of $f$ in the corresponding subscripted subspace, divided by the appropriate scaling factor $2^k$, for $k = 1, 2, \ldots, 9$. We express the above sum simply as

$$
||g * f||^2 = ||S_{11}||^2 |K_{11}|^2 + \cdot \cdot \cdot + ||S_{s,9}||^2 |K_{s,9}|^2,
$$

(2.17)

with the natural identification of terms in (2.16). Referring to the examples of the previous subsection, the correlation coefficients (scaled by 2) of $f_i$ and $f_j$, and also of $f_i * f$ and $f * f$, assuming zero mean real sequences, can be written as

$$
\frac{2 < f_i, f >}{||f_i|| ||f||} = \frac{||f_i||}{||f||} + \frac{||f||}{||f||} - \frac{||f_i - f||^2}{||f_i|| ||f||},
$$

(2.18)

and

$$
\frac{2 < f_i * f, f * f >}{||f_i * f|| ||f * f||} = \frac{||f_i * f||}{||f * f||} + \frac{||f * f||}{||f_i * f||} - \frac{||f_i - f|| * f||^2}{||f_i * f|| ||f * f||}.
$$

(2.19)

The normalized mean-square error in terms of the spectrum is

$$
\frac{||f_i - f||^2}{||f_i|| ||f||} = \frac{||S_{11}||^2 + \cdot \cdot \cdot + ||S_{s,9}||^2}{||f_i|| ||f||},
$$

(2.20)
and

\[ \frac{\| (f_i - f) \ast f \|}{\| f_i - f \|} = \frac{\| S_{11} \|^2 K_{11} K_{34} + \cdots + \| S_{m1} \|^2 K_{34}}{\| f_i - f \|} \tag{2.21} \]

where \( S_{ij} \) refers to the spectrum of \( f_i - f \) and \( K_{ij} \) to that of \( f \). Equations (2.18) to (2.21) describe the relationships between the correlation coefficients before and after WPC convolution.

**Example 2.** We proceed now with a somewhat more general example. Figure 8(a) to (d) shows 16 image resolutions \( f_i \) that are the lowpass (\( f_1 \) to \( f_9 \)) and bandpass (\( f_9 \) to \( f_{16} \)) approximations to the Mandrill image \( f \) at scales \( k = 1, 2, \ldots, 8 \) obtained using the Daubechies wavelets, dB8. The bandpass consists of all three bands at each scale. We investigate the similarity of \( f_i \) to \( f \). Using our earlier highpass example as a guide, we could compare the two signals in a variety of multiresolution bases. We instead utilize the WPC basis: It provides a simple, piecewise constant approximation at different scales; it is local, and projections at different scales are easily obtainable. As in the previous synthetic example, we use the same process for comparison. Lowpass images are convolved and correlated; highpass images are correlated after projection at each scale. These correlation coefficients—referred to as multiresolution correlation, \( \rho_{MR} \)—are compared to the correlation coefficients \( \rho \) obtained directly.

Each subfigure in Figures 8(e) and (f) shows one \( f_i \ast f \), for \( i = 1, 2, \ldots, 16 \) and \( f \ast f \). For easier comparison, \( f_i \ast f \) is scaled so that its first term is the same as \( f \ast f \). We obtain the multiresolution correlation coefficients \( \rho_{MR} \) and correlation coefficient \( \rho \), and for convenience plot them as errors: the normalized multiresolution error (MRE), \( 1 - \text{abs}(\rho_{MR}) \), and the normalized mean square error (MSE), \( 1 - \text{abs}(\rho) \). As expected, and as seen in Figures 9(a) and (b), both the MRE and MSE in the lowpass case (scaled images \( i = 1 \) to 8) increase with increasing scale while remaining high for the bandpass signals. To obtain a measure of discrimination for dissimilar images, we repeat the process, correlating nine other natural images \(^1\) \( g_k \), for \( k = 1, 2, \ldots, 9 \) to \( f \) using the same procedure at scales \( i = 1, 2, \ldots, 16 \). Results are shown in Figures 9(a) and (b). We obtain a correlation coefficient \( 0.95 \leq \text{abs}(\rho_{MR}) \leq 1 \) for images 1 to 6, whereas only images 1 and 2 have \( 0.85 \leq \text{abs}(\rho) \leq 0.95 \). We conclude from \( \rho_{MR} \) that images \( f_i \), for \( i = 1, 2, \ldots, 6 \), have a good match to \( f \) at some scale.

Images \( f_9 \) to \( f_{16} \) in Figures 8(c) and (d) are 8 bandpass images at scales 1 to 8. For each image we adopt the same procedure as for the highpass image of Figure 7(b). That is, for each image \( f_i \) we correlate projections \( f_i \ast h_{HP_j} \) to \( f \ast h_{HP_j} \) for \( j = 1, 2, \ldots, 8 \). For each \( i \), correlation coefficients are calculated for all 8 projections. While a "best match" correlation coefficient could be selected in a number of ways, we select the one corresponding to the known scale of the test image, which also tends to be the highest. As before, we repeat the process for the other nine images. Results are shown in Figure 10. Here too we have a stronger indication of matches than with regular correlation, with \( \text{abs}(\rho_{MR}) \) greater than \( \text{abs}(\rho) \) for all 8 images, and typically, substantially greater. We note also the stronger discrimination capability with WPC convolution: For scaled image 1 to 6 the ratio of the lowest MRE for all the dissimilar images to the MRE for perceptually similar images ranged from 15,000 to 12; for the bandpass images 9 to 14 it ranged from 6.4 to 1.4. Without convolution, the corresponding mean-square errors were 15.4 to 1.3 and 1.4 to 1.3.

It should be noted that in the process of comparing \( f_i \) and \( f \) by convolving first with filter \( f \), the filter merely acts as a weighting function to generate signals that are a sum of projections on subspaces. It is the process of projections, rather than the specific weights of filter \( f \) that helps establish similarity. Hence, many

\(^1\)These images were obtained from the proposed collection of standard images at the Waterloo BragZone.
filters \( f \) that generate a full decomposition and which are also invertible can be used. Furthermore, since \( f_x(-x) \) and \( f(-x) \) are equally valid for determination of similarity, WPC \textit{correlation}, as defined in the next section, may also have been utilized instead of WPC convolution.

3 WPC Group-based Correlation

Classically, correlation plays an important role in comparing signals. Our goal here is to give a formulation of group-based correlation which appears to achieve some of the same recognition features as classical correlation. Mimicking the formulation of discrete cyclic correlation, for an arbitrary finite group we define the \textit{group correlation} of two functions by

\[
f \oplus h = (\tilde{f}) \ast \overline{h}, \quad \text{for all } f, h \in L(X),
\]

(3.1)

where \( \tilde{f} \) is the signal \( f \) in reverse order, \( \overline{h} \) is the complex conjugate of \( h \), and \( \ast \) is group-based convolution.

When the underlying group is a WPC group we shall denote the WPC correlation of \( f \) and \( h \) by \( f \oplus h \) as well (where WPC correlation may be computed efficiently via the convolution algorithm presented earlier). When the WPC group is just a cyclic group, WPC correlation reduces to standard cyclic correlation. When \( f, h \in L(k, n) \), \( \tilde{f} \) is equivalently obtained from \( f \) by reversing it about the independent variable and shifting so as to make a causal sequence on \( Q(k, n) \). When \( f \) and \( h \) are images scanned in the \( Q(1, n) \) quadtree manner, \( \tilde{f} \) is seen to be the image \( f \) reflected about the horizontal line bisecting it. In order to make the WPC correlated output consistent with the location of the image \( f \) we modify our WPC correlation definition slightly: For \( p = f \oplus h \) we let \( g = \overline{p} \) and define \( g = f \oplus h \), giving us WPC correlation as \( g(x) = (f(-x) \ast \overline{h(x)})(-x) \).

We shall see that WPC correlation, which is a natural outgrowth of WPC convolution, extends that notion of similarity, where similarity is measured by a peak matching operation. We see here that data that has for example been cyclical shifted can be correlated to the unshifted data through WPC correlation and determined as “similar.” Moreover, for certain signals, peak values obtained through WPC correlation can be the same as those obtained through standard correlation. As before, we use for our examples images obtained as one-dimensional sequences through quadtree scans. We note that the scale-selective property of WPC convolution extends also to WPC correlation. Hence, as noted earlier, examination of signals for perceptual similarity as considered in Examples 1 and 2 may also be effected through WPC correlation.

In the following subsection we give properties of WPC correlation analogous to Theorem 2.1. WPC correlation is characterized by its group and scale-invairances as also by peak values achieved. The group invariant property of WPC correlation is defined and illustrated by examples, and results are compared with those of linear correlation. The unique capability of WPC correlation in identifying perceptually similar, group-transformed patterns is seen. We also see that WPC correlation can achieve values similar to standard correlation.

3.1 Properties of WPC Correlation

\textbf{Theorem 3.1} Let the finite group \( G \) act transitively on the set \( X \).

(1) Group-based correlation in \( L(X) \) is \textit{bilinear}, but need not be \textit{associative}. 

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(2) Correlation is compatible with the left action of $G$ on $L(X)$:

$$\alpha(f \oplus h) = (\alpha f) \oplus h, \quad \text{for all } f, h \in L(X) \text{ and all } \alpha \in G.$$  

(3) The maximum modulus of $f \oplus h$ (i.e., the largest absolute value of $(f \ast h)(x)$ over all $x \in X$) is equal to the maximum modulus of $\tau(f) \oplus h$ for all $\tau \in G$.

(4) The maximum modulus of the WPC correlation $\tau(f) \oplus f$ is equal to the maximum modulus of the linear correlation of $f$ and $f$ when both signals have support on the first $4^k$ leaves of $Q(k,n)$, and $\tau \in G = Z(k,n)$.

Proof: Properties (1), (2) and (3) follow directly from those of convolution, so we verify (4). When $f$ has support on the first $4^k$ leaves of $Q(k,n)$, then for determining the maximum modulus of WPC correlation $f \oplus f$ we need only compare circular correlation of $f$ and $f$ to their linear correlation. As is the case for circular convolution ([10], pp.548–553), it follows directly that the circular correlation of two finite length sequences is equivalent to the linear correlation of the two sequences followed by aliasing. In the case here, since both the length and period of $f$ is the same ($4^k$), it is easily seen that in periodic correlation, all points but one are corrupted by aliasing. The uncorrupted point is at 0, which contains the maximum modulus of linear correlation. Furthermore, invoking property (3), we also see that the maximum modulus of a group transformed version of $f$ WPC correlated with $f$ is the same as that of the linear correlation of $f$ and $f$. This completes the proof.

For the remainder of this subsection we assume $G = Z(1,9)$ acting on the tree $Q(1,9)$, and signals $f$ and $h$ lie in $L(1,9)$. For a geometrical visualization of WPC correlation we address, as before, correlation with $f_c$ and the delta functions $h_i$, for $i = 0, 1, \ldots, 9$. Based on the definition of correlation involving a time-reversal of both $f$ and $f \ast h$, the correlation properties evolve in a natural way from the convolution properties of Theorem 2.7. Hence, for filter $h_0$ we have

$$f_c \oplus h_0 = f_c = g_1,$$

$$f_c \oplus (\alpha_0^{(8)})^k(h_0) = (\alpha_0^{(8)})^{-k} f_c = (\alpha_0^{(8)})^{-k} g_1, \quad \text{for } k = 0, 1, 2, 3.$$  

For filter $h_2$ we have

$$f_c \oplus h_2 = f_c \ast \alpha_0^{(T)}(h_0) = (\alpha_0^{(T)})^{-1}(f_c h_0) \equiv g_2,$$

$$f_c \oplus (\alpha_0^{(8)})^k(h_2) = f_c \ast \alpha_0^{(8)}(\alpha_0^{(8)})^k(h_0) = (\alpha_0^{(8)})^{-1}(f_c h_0) \equiv g_2,$$

$$f_c \oplus (\alpha_0^{(T)})^k(h_2) = (\alpha_0^{(T)})^{-k} (g_2), \quad \text{for } k = 0, 1, 2, 3.$$  

Similarly, we see that correlating signal $f_c$ with the other unit impulse filters $h_i$ and their extensions, for $i = 3, 4, \ldots, 9$, results in an output $g_i$ consisting of averages $f_{c_{i-10}}$, located in quadrants $Q_{i-9}$. As before, the only group transformations on $h_i$ that we need consider are operations at levels $9 - i$. Accordingly, for filter $h_9$ we have

$$f_c \oplus h_0 = f_c \ast \alpha_0^{(0)}(h_0) = (\alpha_0^{(0)})^{-1}(f_c h_0) \equiv g_9,$$

$$f_c \oplus (\alpha_0^{(0)})^k(h_0) = (\alpha_0^{(0)})^{-k} (g_9), \quad \text{for } k = 0, 1, 2, 3.$$
3.2 WPC Correlation and Group Transformed Images

We now look to its physical interpretation of WPC correlation and its relationship to linear correlation in the context of similarity of signals. Linear correlation of two signals $f$ and $g$ essentially performs a text-string matching operation, the matched filter exemplifying this process in the classic problem of detection of a known signal in noise. Correlation then is measured by its maximum value, which occurs when two sequences or patterns match exactly; however, linear correlation perform is unable to classify similar, sequences that may have, for example, elements interchanged, or may be circularly rotated (or group transformed) versions of each other. As a simple example, the 64-point sequence $f = [7, 2, 6, 5, 0, 0, \ldots, 0]$ when linearly correlated with itself gives a maximum value of 114; whereas $f_1 = [0, 0, 0, 0, 5, 7, 2, 6, 0, 0, \ldots, 0]$—a slightly modified version of $f$—when correlated with $f$, gives a maximum value of 91. With the $Q(1,3)$ tree, WPC correlations $f \oplus f$ and $f_1 \oplus f$ both show a maximum of 114, the same as that obtained with the linear correlation of $f$ and $f$. (Note that WPC convolution does not achieve that maximum, nor does cyclic convolution.)

Sequence $f_1$ has the form $\tau(f)$, i.e., is a $Q(1,3)$ group transformed version of $f$. By property (4) of Theorem 3.1, such transformations preserve the maximum achieved in linear correlation of $f_1$ and $f$ when the two signals have the appropriate support. Hence we observe (as follows from the properties) that WPC correlation of $f_1$ and $f$ achieves the maximum value obtained with linear correlation when $f$ has support on the first $4^k$ leaves of the $Q(k,n)$ tree and when $f_1$ is a group transformed version of $f$. When WPC correlation of two signals $f \oplus h$ is employed for matching using peak detection, it should be noted (keeping in mind the non-commutativity of correlation) that it is $h$ that forms the filter or template and $f$ the noisy or transformed signal that we attempt to match. Since segments of the convolving filter $h$ filter different scales of the input signal, it is appropriate to use the transformed or noisy signal as the input signal $f$ and the template as the feature selecting filter $h$.

We generalize to a slightly more complex example wherein we desire to match a specific 4 × 4 template image $h$ to group transformations $f$ of that image. All patterns are located in a 256 × 256 array. Accordingly, we correlate the transformed images $f$ with the template $h$. Here we assume that the WPC group is $Z(2,4)$ acting on the tree $Q(2,4)$, and that the template image is supported on the first $4^2$ leaves of the tree, i.e., the leftmost subtree in $Q(2,4)$. At each level of the tree the image is subdivided into a succession of 4 × 4 grids, with a clockwise spiral ordering of the blocks. Note that with this ordering, the effect of a cyclic shift is to rotate the blocks and spiral them towards the center, with the "central" block moved to the upper left corner (so this is not a planar symmetry of the image). We consider sixteen 4 × 4 images embedded in 256 × 256 arrays. For easier visualization Figure 11 shows all 16 binary images in their actual location in one array. The upper lefthand corner contains the template "T" image which represents the filter $h$ and which also represents the first of the images to be detected ($f_1$). The other 15 images, $f_2$ through $f_{16}$, constitute the remainder of the transformed image set. Images $f_2$, $f_3$ and $f_4$, which are three rotated (in multiples of 90°) versions of $h$, are shown below the filter image. The second column contains images $f_5$ and $f_7$, and $f_8$, which are rotated versions of $h$ at angles of 45°, 135°, 225° and 315° respectively. The third and fourth columns contain eight group transformations of $h$. That is, $f_9, \ldots, f_{16}$ are $a_k^h$ where $k = 1, 2, 3, 4, 5, 9, 11, 13$, and where $a_0 \in S_{16}$ is the group element of order 16 cycling the first 16 leaves of $Q(2,4)$. Note that other than the filter image $h$, which must exist in the top lefthand corner, all other rotated and group transformed images are translated to lie in any 4 × 4 quadrant naturally created by the $Q(2,4)$ subdivision.
We correlate $f_i$ with $h$ for $i = 1, 2, \ldots, 16$ using both WPC correlation and linear correlation. Since the template image "T" contained in $h$ is entirely defined on the leftmost subtree of $Q(2, 4)$, we know that WPC correlation of $f_i$ and $h$ will achieve the maximum value of $12$, which is likewise reached with both linear 1-D correlation and also 2-D correlation. As expected, 2-D correlation is unable to identify any of the rotated or group transformed images, in that the peak value of $12$ is reached only for the case of $f_i$ correlated with $h$. On the other hand, WPC correlation, as expected, also achieves the peak value $12$ for $i = 9, 10, \ldots, 16$, corresponding to the cases of group transformed images. For comparison purposes we also WPC correlate using the $Q(1, 8)$ decomposition. For the $Q(1, 8)$ tree, WPC correlation identifies as similar (same peak values), rotated images $f_2$, $f_3$ and $f_4$ since they correspond to group transformations of $Q(1, 8)$; $f_9$ through $f_{16}$ are not so identified. As expected, the maximum value of $12$ is not reached since the $h$ is defined on more than the leaves of the leftmost subtree of $Q(1, 8)$. Numerical results are given in Table I. Note that figure "T" supported on the first $4^2$ leaves of $Q(2, 4)$ and scanned in a clockwise spiral ordering generates a symmetric signal. Hence results for WPC convolution and correlation based on the $Q(2, 4)$ tree, involving $f_i$ and $f_9$ to $f_{16}$, yield the same maximum of $12$. Other similar values are coincidental, and stem from the utilization of binary images.

4 Conclusion

The algebraic structure of a group theoretic approach gives the advantage of a natural collection of (noncommutative) convolution operators. When this is applied to the quadtree indexing of a digitized image, these convolution operators have potential for application in signal processing. The main contribution of this paper and its predecessor [6] has been to establish some of the basic properties of finite group-based signal processing—in particular, WPC convolution and correlation—and illustrate their application through some examples. Most of the exposition here has been confined to the transform generated from the WPC group $Z(k, n)$ acting on the quadtree $Q(k, n)$. This group action provides a natural multiresolution analysis of $2^{kn} \times 2^{kn}$ images which generalizes the DFT and circular convolution.

An explicit formula for WPC convolution has been derived. Properties of WPC convolution and correlation have been developed to aid their use for this new class of linear, noncommutative filters. The basic utility of the filters in frequency as well as in a scale-selective mode were given. We have seen that the process of WPC filtering establishes a multiresolution decomposition of a signal where the signal is represented as a sum of projections on all the subspaces. This corresponds to a transformation of the signal, where it is represented as a sum of signals that are piecewise constant over intervals of length $4^k$, for $k = 0, 1, \ldots, n - 1$. This representation was utilized in assessing the capability of WPC filtering in determining similarity of scaled images. Signals were first transformed by WPC convolution and then the standard correlation measure applied to establish similarity. It was seen that lowpass perceptually similar test images had markedly higher correlations after WPC convolution than that before the transformation. Bandpass images also showed higher correlations when both the template and test images were correlated at individual scales. In addition, the discrimination capability for dissimilar images was considerably greater than that obtained with standard correlation. This was illustrated through an example using 16 scaled versions of one image and comparing them to the unscaled (original) image as well as nine other (dissimilar) images.

WPC correlation was used to identify images that were group transformed versions of a prototype image. For these images, and with the appropriate selection of quadtree decomposition, WPC correlation was seen to
achieve the same peak value as standard correlation obtains for two identical images.

4.1 Open Research

This paper has explored some of the signal processing implications of spectral analysis of the WPC group associated with the \( Q(k, n) \) quadtree. Applications were confined to two-dimensional images reduced to one-dimensional signals through a quadtree scan. At this early stage of development and application of WPC group theory, a host of open problems exist. A few of them are as follows:

1. Further investigations of WPC convolution and correlation: We have seen that transforming signals using WPC convolution can be useful in establishing similarity of certain types of images, while still maintaining a discrimination capability. A quantitative treatment of similarity determination, classification accuracy, and scalability using WPC convolution is needed. Common image distortions such as rotation, scale, and translation need to be considered. WPC correlation should be investigated to determine if both its peak preserving and pattern preserving properties can aid similarity determination.

2. Applications of WPC group spectral properties to multiscale analysis: General multiscale analysis often uses the spectrum at various scales in applications such as pattern recognition, noise removal, image registration, etc. The complex WPC group spectrum makes available amplitude and, more importantly phase, at all scales, as also a group invariance and a localized transform property. Consideration of these properties to applications should provide a rich area for investigation. For instance, like the unnormalized Haar transform [16], the unnormalized WPC transform can also be looked upon as a multiscale analysis tool for Poisson processes. How might the additional group invariant structure of the WPC transform and WPC convolution and correlation contribute to the modeling and estimation of Poisson processes?

3. Noise filtering: What process does the WPC transform diagonalize and what is its practical significance? For signal and noise filtering or prediction problems, what is the MSE in terms of the WPC convolution, and what conditions on the convolution ensure that it is the smallest?

4. Extensions to 2-D: Given the one-dimensional unitary WPC transform, a two-dimensional separable unitary transform can be defined; however, its group theoretic implications are not presently clear. Does there exist such a theory? If so, what are its implications, especially those analogous to the one-dimensional case?
5 Appendix A — Proof of Theorem 2.6

Parts (2) and (3) of the theorem are direct consequences of parts (1) and (4) in Theorem 2.1, so we concentrate on verifying (1). From formula (2.2) and the definitions of \( f_0 \) and \( h_i \) we have

\[
(f_0 \ast h_i)(x_p) = \frac{X}{|G|} \sum_{\beta \in G} f_0(\beta x_0) h_i(\beta^{-1} x_p) = \frac{X}{|G|} \sum_{\beta \in G_0} h_i(\beta^{-1} x_p)
\]

\[
= \frac{X}{|G|} \sum_{\beta \in G_0 \atop \beta x_i = x_p} 1 = \frac{X}{|G|} \cdot |\{ \beta \in G_0 \mid \beta x_i = x_p \}|. \tag{A1}
\]

Now \( G_0 \) consists of those symmetries of the tree \( Q(1, n) \) (as depicted in Figure 7 of [6] for \( n = 9 \)) that fix the leftmost path from node 0 down to the zeroth leaf, \( x_0 \). The subgroup \( G_0 \) therefore acts as symmetries of the three trees of type \( Q(1, n-1) \) descending from nodes 1, 2 and 3 at level 1. It also acts as symmetries of the three trees of type \( Q(1, n-2) \) descending from nodes 1, 2 and 3 at level 2, and so on. The symmetries of each of these subtrees may be specified independently, so that \( G_0 \) is the direct product of the full WPC groups of symmetries of all these subtrees:

\[
G_0 \cong Z(1, n-1)^3 \times Z(1, n-2)^3 \times \cdots \times Z(1, 1)^3, \tag{A2}
\]

where \( Z(1, m)^3 \) denotes the direct product \( Z(1, m) \times Z(1, m) \times Z(1, m) \). From this perspective the number of elements \( \beta \) in \( G_0 \) such that \( \beta x_i = x_p \) can easily be computed as follows. Let \( s \) be the smallest integer such that both \( x_i \) and \( x_p \) descend from a common node at level \( s \) other than the zeroth node at level \( s \); if no such common ancestor exists, then \( (f_0 \ast h_i)(x_p) = 0 \) because there is no element \( \beta \) in \( G_0 \) with \( \beta x_i = x_p \) (i.e., \( x_i \) and \( x_p \) lie in different orbits of \( G_0 \) acting on the leaves). In other words, we consider the unique largest subtree containing both \( x_i \) and \( x_p \) whose root node is the \( d^{th} \) node, \( \nu_d^0 \), at level \( s \), for some \( d \geq 1 \), if such exists. Then the symmetry group \( Z(1, n-s) \) of the subtree descending from node \( \nu_d^0 \) is one of the direct factors of \( G_0 \) described in (A2). By the remark following (3.6) in [6], in the group \( Z(1, n-s) \) the number of elements that send one leaf of \( Q(1, n-s) \) to any other is the same as the number of elements that fix a leaf, namely \( |Z(1, n-s)|/4^{n-s} \). Since all the remaining direct factors of \( G_0 \) permute sets of leaves that are disjoint from the block of leaves descending from \( \nu_d^0 \), any element of \( Z(1, n-s) \) that maps \( x_i \) to \( x_p \) may be multiplied by any elements from the remaining direct factors to result in another element of \( G_0 \) that still sends \( x_i \) to \( x_p \). It therefore follows that the number of elements in the set \( \{ \beta \in G_0 \mid \beta x_i = x_p \} \) is \( |G_0|/4^{n-s} \) (again assuming \( x_i \) and \( x_p \) lie in a common subtree not rooted at a zeroth node). Since \( |X| = |G|/|G_0| \), the formula in part (1) of the theorem is seen to hold. This completes the proof.
References


Figure 1: Average of $f_C$ over nine quadrants

Figure 2: Unit impulse filters and extensions
Figure 3: Scale selective lowpass filters and spectra

Figure 4: Scale selective bandpass filters and spectra
Figure 5: WPC convolution of image f with filters h

Figure 6: Canonic and equivalent filters
Figure 7: WPC convolution using WP filtered images
(a) Lowpass images $f_1 \rightarrow f_4$ at scales 1-4  
(b) Lowpass images $f_5 \rightarrow f_8$ at scales 5-8  
(c) Bandpass images $f_9 \rightarrow f_{12}$ at scales 1-4 
(d) Bandpass images $f_{13} \rightarrow f_{16}$ at scales 5-8  
(e) $f_1 \ast f$ (lowpass) and $f \ast f$  
(f) $f_3 \ast f$ (bandpass) and $f \ast h$  

Figure 8: WPC convolution using dB8 filtered images
Figure 9: Comparison of errors with and without WPC convolution

Figure 10: Correlation coefficient error after WPC convolution (bandpass images)
Figure 11: Image “T” and 15 rotated and group transformed images

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<td>12</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>9</td>
</tr>
</tbody>
</table>

Table 1: Comparison of WPC correlation and convolution with linear correlation