Partial Derivatives: Let $z = f(x, y)$. Then the partial derivative of $f$ with respect to $x$ is the derivative when $y$ is held constant. It is written $\frac{\partial f}{\partial x}$. The partial of $f$ with respect to $y$ is the derivative when $x$ is held constant. It is written $\frac{\partial f}{\partial y}$.

Note: $\frac{\partial f}{\partial x}(a, b)$ is the slope of the line tangent to the curve $z = f(x, y)$ at the point $(a, b)$ moving parallel to the $x$-axis. $\frac{\partial f}{\partial y}(a, b)$ is the slope of the line tangent to the curve $z = f(x, y)$ at the point $(a, b)$ moving parallel to the $y$-axis. We do not consider slopes of tangent lines moving in any other direction.

**Example:** Let $P(s, l)$ be the profit in hundreds of dollars if I sell $s$ snow blowers and $l$ lawn mowers. Then $\frac{\delta P}{\delta s} = P_s$ would tell me how changing the number of snow blowers sold when the numbers of lawn mowers does not change that changes the profit. Likewise, $\frac{\delta P}{\delta l} = P_l$ tells me how profits change if the number of lawn mowers changes but the number of snow blowers is constant.

**Example:** $f(x, y) = 2x^2 + 3y - 4$
Find $f_x(-1, 1)$ and $f_y$.
$f_x(x, y) = 4x + 0$
$f_x(-1, 1) = -4$
$f_y = 3$

**Example:** $f(x, y) = xe^y + 2x^2 y^3$
Find: $\frac{\delta f}{\delta x}$ and $\frac{\delta f}{\delta y}$
\[
\frac{\delta f}{\delta x} = e^y + 2x \left(2y^3\right) = e^y + 4xy^3
\]
\[
\frac{\delta f}{\delta y} = f_y = xe^y + \left(2x^2\right) \left(3y^2\right) = xe^y + 6x^2y^2
\]

**Second Order Partial Derivatives:**
First-order partial derivatives
\[
\frac{\delta f}{\delta x} = f_x \quad \text{and} \quad \frac{\delta f}{\delta y} = f_y
\]

We can take the derivative of each of these:
\[
\frac{\delta}{\delta x} \left[ \frac{\delta f}{\delta x} \right] = \frac{\delta^2 f}{\delta x^2} \quad \text{and} \quad \frac{\delta}{\delta y} \left[ \frac{\delta f}{\delta x} \right] = \frac{\delta^2 f}{\delta y \delta x}
\]
\[
\frac{\delta}{\delta y} \left[ f_x \right] = f_{xx} \quad \frac{\delta}{\delta y} \left[ f_y \right] = f_{xy}
\]
also,
\[
\frac{\delta}{\delta x} \left[ \frac{\delta f}{\delta y} \right] = \frac{\delta^2 f}{\delta y^2} \quad \frac{\delta}{\delta x} \left[ \frac{\delta f}{\delta y} \right] = \frac{\delta^2 f}{\delta x \delta y}
\]
\[
\frac{\delta}{\delta y} \left[ f_y \right] = f_{yy} \quad \frac{\delta}{\delta x} \left[ f_y \right] = f_{yx}
\]

**Example:** \( f(x, y) = x^3 + 3x^2y - 4y^4 \)