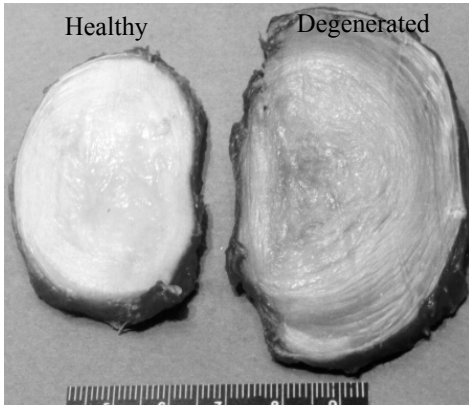
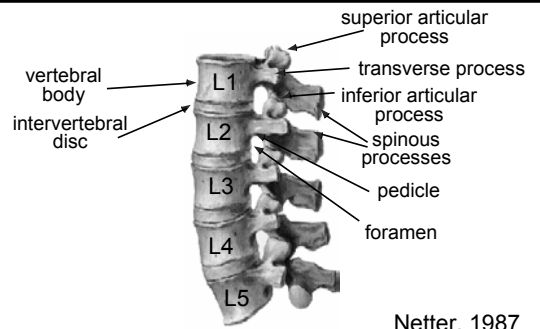


Linear Viscoelasticity and the Relaxation Spectrum

James Iatridis, PhD
Dept. of Mechanical Engineering
UVM

Lumbar Spine



Viscoelasticity

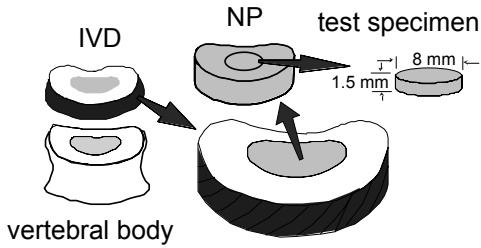
- Our bodies are mostly water
 - it just makes sense that water affects the mechanical behavior of biological soft-tissues
- Viscoelasticity
 - Flow independent viscoelasticity
 - polymeric matrix molecules reorient ... at their own pace
 - No volume changes
 - Flow dependent (Biphasic) viscoelasticity:
 - Water is forced out of the tissue like a sponge
 - Very small pores so this takes time
 - Requires volume changes

The Viscoelastic Shear Behavior of the Human Lumbar Nucleus Pulposus

Objectives

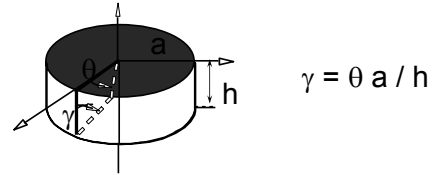
- Study the intrinsic viscoelastic behavior of the NP in shear
- Determine a constitutive relationship capable of describing this viscoelastic behavior

Specimen Preparation



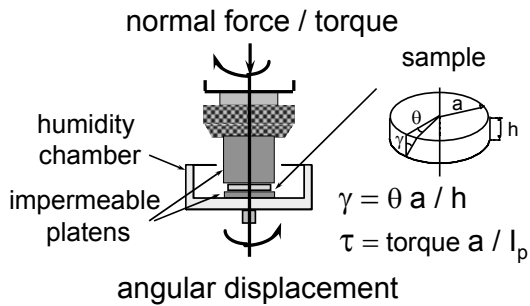
Mechanical Testing

- Torsional shear strain (γ)

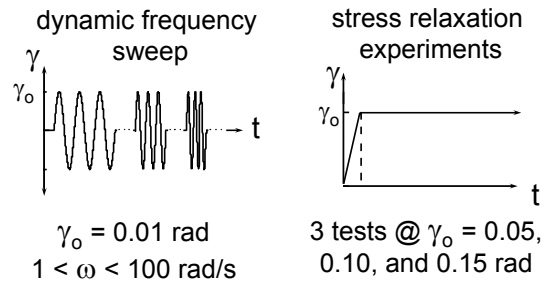


- No volumetric changes --> Negligible flow-dependent effects

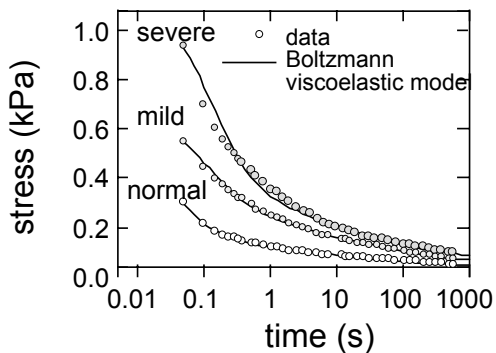
Shear Testing Apparatus



Shear Testing Protocol



Stress-Relaxation Experiment



Viscoelastic modeling

Elastic Solid & Viscous Fluid

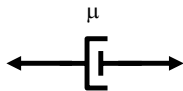
- Elastic solid

$$\tau = G\gamma$$



- Viscous fluid

$$\tau = \mu \dot{\gamma}$$



Viscoelastic Models

Maxwell Fluid: 2 constants

- Stress is a function of strain-rate

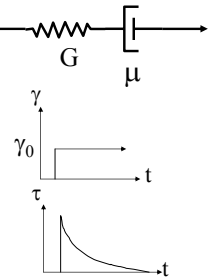
$$\tau + \lambda \dot{\tau} = \mu \dot{\gamma}$$

where $\lambda = \mu/G$ and is the time constant

- Stress -relaxation function

$$\tau(t)/\gamma_0 = G(t) = (\mu/\lambda)\exp(-t/\lambda)$$

Relaxation function defined by viscosity μ and relaxation time λ



3 parameter solid

- Stress is a function of strain and strain-rate

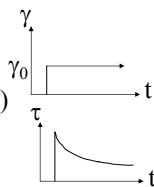
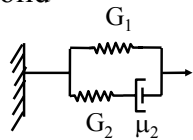
$$\lambda_2 \dot{\tau} + \tau = (\mu_2 + \lambda_2 G_1) \dot{\gamma} + G_1 \gamma$$

- Relaxation Function

$$\tau(t)/\gamma_0 = G(t) = G_1 + (\mu_2/\lambda_2)\exp(-t/\lambda_2)$$

The relaxation function is defined

by a single time constant, $\lambda = \mu/G$



Generalized Maxwell Model

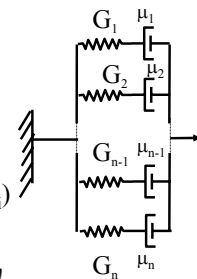
2*n parameters

- Consists of 'n' Maxwell units in parallel

- Stress -relaxation function

$$G(t) = \sum (\mu_i/\lambda_i) \exp(-t/\lambda_i)$$

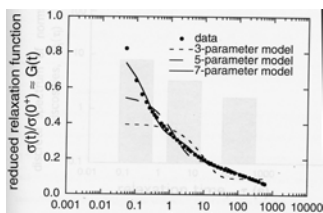
A relaxation function with DISCRETE relaxation times!



Experimental data for human nucleus pulposus

- Curve fit with generalized maxwell model to obtain magnitudes of the time constants

Good description of behavior!



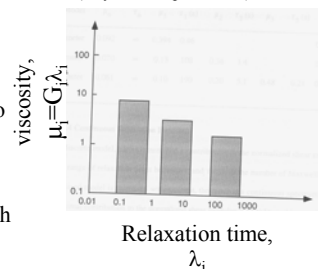
Discrete relaxation spectrum

- 7 material parameters

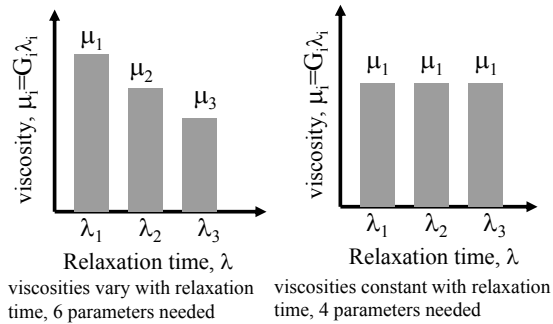
- 3 viscosities
- 3 time constants
- 1 equilibrium shear modulus

- Each bar corresponds to a single relaxation time and viscosity
- Note that amplitude of viscosities decrease with relaxation time

Relaxation spectrum (only shows 6 parameters)



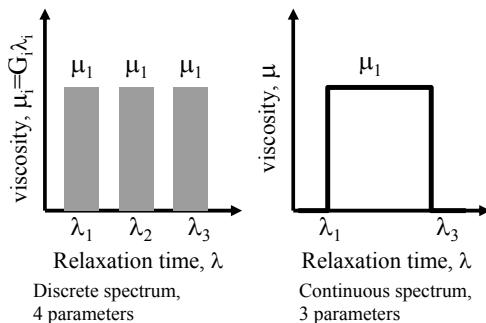
Discrete relaxation spectrum (2)



Discrete relaxation spectrum (3)

- If a material has viscosities that are not a function of relaxation time, it is called a strain rate insensitive material
- If a material has viscosities that are a function of relaxation time, it is called a strain rate sensitive material

Discrete & Continuous Relaxation Spectra



Continuous Relaxation function

- An integral formulation for a continuous spectrum of relaxation times with constant amplitude

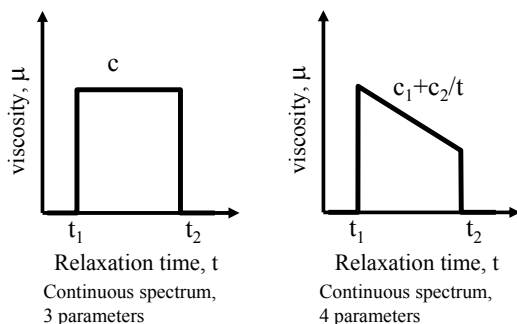
$$G(t) = 1 + \int_0^{\infty} S(t') \exp(-t/t') dt'$$

$$S(t) = \begin{cases} c/t & t_1 < t < t_2 \\ 0 & \text{otherwise} \end{cases}$$

Fung, 1981

Only 3 constants: c, t_1, t_2
these are similar to μ_1, λ_1 and λ_3

Continuous Relaxation Spectra



Continuous relaxation spectrum viscosities that decrease with relaxation time

Integral formulation with viscosities that decrease with relaxation time

$$G(t) = 1 + \int_0^{\infty} S(t') \exp(-t/t') dt' \quad \text{Fung, 1981}$$

Variable amplitude spectrum

$$S(t) = \begin{cases} (c_1 + c_2/t)/t & t_1 < t < t_2 \\ 0 & \text{otherwise} \end{cases}$$

latridis et al., 1997

Reduced relaxation function

- Normalized by the value at $t=0$
- For a discrete relaxation spectrum, we have:

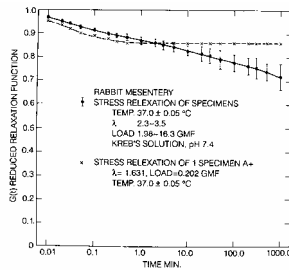
$$G(t) = \frac{\sum C_i e^{-t/\lambda_i}}{\sum C_i}$$

Reduced relaxation function

- 2 important points
 - if an experiment is cut off prematurely, one may mistakenly arrive at the limiting value $G(\infty)$, corresponding to $\lambda_i = \infty$
 - the relaxation times, λ_i should not be interpreted literally without realization that representation of empirical data by a sum of exponentials is a non-unique process

Defining appropriate equilibrium time

- Stress relaxation for an individual specimen ended at 1 min, however, for the average it went beyond 1000 min
- Fung, Fig7.5:4



Non-uniqueness of exponentials

- These 3 exponentials all fit a certain set of experimental data equally well for x between 0 and 1. According to Lanczos (1956) and taken from Fung, p. 280.

$$f(x) = 2.202e^{-4.45x} + 0.305e^{-1.58x},$$

$$f(x) = 0.0951e^{-x} + 0.8607e^{-3x} + 1.5576e^{-5x},$$

$$f(x) = 0.041e^{-0.5x} + 0.79e^{-2.73x} + 1.68e^{-4.96x}.$$

Reduced relaxation function with variable amplitude spectrum

Reduced relaxation function

reduced means normalized by the value at $t=0$

$$G(t) = \frac{1 + \int_0^{\infty} S(t') \exp(-t/t') dt'}{1 + \int_0^{\infty} S(t') dt'} \quad \text{Fung, 1981}$$

Variable amplitude spectrum

$$S(t) = \begin{cases} (c_1 + c_2/t)/t & t_1 < t < t_2 \\ 0 & \text{otherwise} \end{cases}$$

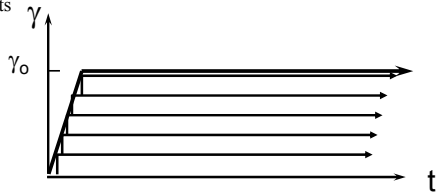
latridis et al., 1997

Integral formulation for linear viscoelastic models

- Now that we know how to describe a step stress-relaxation experiment, we are ready to figure out how to describe any type of loading experiment

Ramp stress-relaxation experiment

- A ramp stress relaxation experiment can be considered to be comprised of a series of infinitesimally small step stress relaxation experiments



- Any (displacement controlled) experiment can be considered to be comprised of a series of step stress relaxation summed together appropriately

Boltzmann Linear Viscoelasticity Model (Integral formulation for stress response)

- Boltzmann superposition: Integral formulation for loading modes other than step strain input
- That is the superposition of infinite number of step stress relaxation experiments

$$\tau(t) = \int_{-\infty}^t G(t-t') \dot{\gamma} dt'$$

$\tau(t)$ = shear stress

$\gamma(t)$ = shear strain

$G(t)$ = reduced relaxation function

G = instantaneous shear modulus

Boltzmann Linear Viscoelasticity Integral Formulation

- If the motion starts at $t=0$ and $\tau = \dot{\gamma} = 0$ for $t < 0$, we have the following (more practical) relationship:

$$\tau(t) = G(0)\dot{\gamma}(t) + \int_0^t \dot{G}(t-t')\dot{\gamma}(t') dt'$$

but $G(0)=1$

- therefore, the shear stress at time t is equal to the instantaneous stress response decreased by an amount depending on the past history because dG/dt' is usually negative

Boltzmann Linear Viscoelasticity Model (for strain response)

- Boltzmann superposition can also be used for creep responses.... That is the superposition of infinite number of step creep experiments

$$\gamma(t) = \int_{-\infty}^t J(t-t') \dot{\tau} dt'$$

$\tau(t)$ = shear stress

$\gamma(t)$ = shear strain

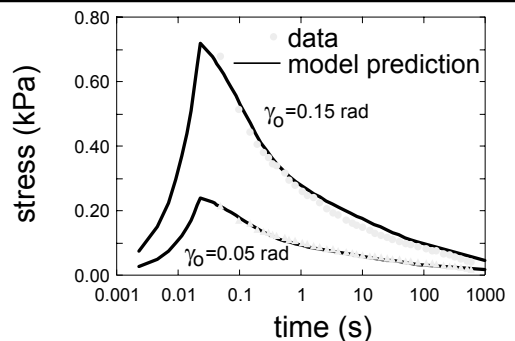
$J(t)$ = reduced creep function

J = instantaneous shear compliance

Model predictions

- Use the Boltzmann linear viscoelasticity model (for stress response) with variable amplitude and continuous relaxation spectrum to describe and predict material behaviors

Stress Relaxation



Future Directions

- We now have a good understanding of different relaxation spectra and how they are used to describe flow-independent viscoelastic behaviors.
- The next topic will be to apply the linear viscoelastic models to experimental data.