The Existence of Referee Squares

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Abstract

In this short note, we prove a conjecture of Anderson, Hamilton and Hilton [1] on the existence of referee squares.

1 Introduction

Let $n$ be an odd integer. A referee square of order $n$ is an $n \times n$ array $R$ based on $S = \{1, 2, \ldots, n\}$ such that

1. each cell is either empty or contains an unordered pair of distinct symbols on $S$,

2. each $i \in S$ occurs precisely once in each row (except the $i$th) and in each column (except $i$th column), and does not occur in the $i$th row and $i$th column,
3. each unordered pair of distinct elements of $S$ occurs in exactly one cell of $R,$

4. the main diagonal cells are non-empty.

Note the close relationship between referee squares and the more well-known object, the Room square (see [4] for a survey of Room squares).

Referee squares were first introduced in [1] where it was conjectured that they exist for all odd orders $n \geq 3$ with $n \neq 5$. In 1998 Y.S. Liaw [5] made progress towards this conjecture by proving the following:

**Theorem 1.1 (Liaw [5])** There exists a referee square of order $n$ for any odd composite integer $n$ and for all $3 \leq n \leq 47$ except that there is no referee square of order 5.

In this short note, we solve the existence problem completely by proving the following result.

**Theorem 1.2** If $n \geq 3$, $n \neq 5$ and $n$ odd, there exists a referee square of order $n$.

2 **Constructions**

The main recursive construction uses frames. But in order to apply the recursion, a few small orders are needed first. In order to obtain these we use variant of a strong starter.

A **strong referee starter $S$** of order $v$ in $\mathbb{Z}_v$ is a set of unordered pairs $S = \{\{s_i, t_i\} : 1 \leq i \leq (v-1)/2\}$ which satisfies the following properties:

1. $\{s_i : 1 \leq i \leq (v-1)/2\} \cup \{t_i : 1 \leq i \leq (v-1)/2\} = \{1, 2, \ldots, n-1\}$

2. $\{\pm(s_i - t_i) : 1 \leq i \leq (v-1)/2\} = \mathbb{Z}_v \setminus \{0\} = \{1, 2, \ldots, n-1\}$

3. if $s_i + t_i \equiv s_j + t_j \pmod{v}$, then $i = j$

4. for some $i$, $s_i + t_i \equiv 0 \pmod{v}$. 
In [5] (Theorem 2.1) it is proven that a referee square of order \( v \) exists if there exists two starters with distinct distances and which contains a zero distance. It is easy to show that given a strong referee starter \( S \), the two starters \( S \) and \(-S\) satisfy this property. Hence the existence of a strong referee starter of order \( v \), implies the existence of a referee square of order \( v \).

**Lemma 2.1** There exist referee squares of orders \( n = 53, 59, 61, 79 \).

**Proof:** For each of these orders we give a strong referee starter. In each case, the pair with difference 1 has sum congruent to zero modulo \( n \).

\[ n = 53 \]
\[
26,27 \ 43,45 \ 34,37 \ 20,24 \ 8,13 \ 5,11 \ 3,10 \ 6,14 \ 40,49 \ 47,4 \ 18,29 \ 30,42 \ 41,1 \ 9,23 \\
17,32 \ 19,35 \ 38,2 \ 33,51 \ 12,31 \ 16,36 \ 39,7 \ 46,15 \ 21,44 \ 28,52 \ 25,50 \ 22,48
\]

\[ n = 59 \]
\[
29,30 \ 12,14 \ 1,4 \ 27,31 \ 45,50 \ 49,55 \ 13,20 \ 40,48 \ 34,43 \ 56,7 \ 36,47 \ 41,53 \ 22,35 \\
18,32 \ 46,2 \ 9,25 \ 11,28 \ 33,51 \ 57,17 \ 6,26 \ 3,24 \ 52,15 \ 19,42 \ 58,23 \ 39,5 \ 54,21 \ 10,37 \\
16,44 \ 38,8
\]

\[ n = 61 \]
\[
30,31 \ 32,34 \ 6,9 \ 4,8 \ 40,45 \ 52,58 \ 37,44 \ 47,55 \ 24,33 \ 26,36 \ 28,39 \ 59,10 \ 12,25 \ 50,3 \\
14,29 \ 7,23 \ 5,22 \ 60,17 \ 35,54 \ 43,2 \ 41,1 \ 57,18 \ 53,15 \ 48,11 \ 13,38 \ 16,42 \ 19,46 \ 21,49 \\
27,56 \ 51,20
\]

\[ n = 79 \]
\[
39,40 \ 74,76 \ 58,61 \ 37,41 \ 44,49 \ 23,29 \ 68,75 \ 6,14 \ 22,31 \ 9,19 \ 59,70 \ 36,48 \ 7,20 \\
66,1 \ 2,17 \ 26,42 \ 60,77 \ 51,69 \ 63,3 \ 5,25 \ 11,32 \ 12,34 \ 27,50 \ 38,62 \ 28,53 \ 45,71 \\
30,57 \ 55,4 \ 43,72 \ 73,24 \ 15,46 \ 35,67 \ 64,18 \ 78,33 \ 65,21 \ 56,13 \ 52,10 \ 16,54 \ 8,47
\]

Now for the frame constructions. Let \( S \) be a set, and let \( \{S_1, S_2, \ldots, S_n\} \) be a partition of \( S \). An \( \{S_1, S_2, \ldots, S_n\} \)-Room frame is an \( |S| \times |S| \) array, \( F \), indexed by \( S \), that satisfies the following properties:
1. Every cell of $F$ either is empty or contains an unordered pair of distinct symbols of $S$.

2. The subarrays $S_i \times S_i$ are empty, for $1 \leq i \leq n$ (these subarrays are referred to as holes).

3. Each symbol $x \not\in S_i$ occurs in row (or column) $s$, for any $s \in S_i$.

4. The pairs in $F$ are those $\{s, t\}$, where $(s, t) \in (S \times S) \setminus \cup_{i=1}^{n}(S_i \times S_i)$.

As is usually done in the literature, we refer to a Room frame simply as a frame. The type of a frame $F$ is defined to be the multiset $\{|S_i| : 1 \leq i \leq n\}$. We usually use an “exponential” notation to describe types: a type $t_1^{u_1} t_2^{u_2} \ldots t_k^{u_k}$ denotes $u_i$ occurrences of holes of size $t_i$, $1 \leq i \leq k$.

**Theorem 2.2** Suppose there exists a frame of type $t_1^{u_1} t_2^{u_2} \ldots t_k^{u_k}$, if there exists a referee square for $t_i$, $1 \leq i \leq k$, then there exists a referee square of order $\sum_{i=1}^{k} t_i u_i$.

**Proof:** Fill in each hole $S_i \times S_i$ of side $t \times t$ by putting in a referee square of order $t$ containing the symbols of $S_i$. \hfill \square

Let $K$ be a set of positive integers. A group divisible design $K$-GDD is a triple $(X, \mathcal{G}, \mathcal{A})$ where

1. $X$ is a finite set of points,

2. $\mathcal{G} = \{S : \infty \leq \} \leq \}$ is a set of subsets of $X$, called groups, which partition $X$,

3. $\mathcal{A}$ is a collection of subsets of $X$ with sizes from $K$, called blocks, such that every pair of points from distinct groups occurs in exactly 1 block, and

4. no pair of points belonging to a group occurs in any block.

The type of a GDD is defined to be the multiset $\{|S_i| : 1 \leq i \leq n\}$. Again the “exponential” notation is used to describe types: a type $t_1^{u_1} t_2^{u_2} \ldots t_k^{u_k}$ denotes $u_i$ occurrences of groups of size $t_i$, $1 \leq i \leq k$. The following is the Fundamental Frame Construction (see [4]).
**Theorem 2.3** Let $(X, G, A)$ be a GDD, and let $w : X \to \mathbb{Z}^+ \cup \{0\}$ be a weight function on $X$. Suppose that for each block $A \in A$, there exists a frame of type $\{w(x) : x \in A\}$. Then there is a frame of type $\{\sum_{x \in G_i} w(x) : G_i \in G\}$.

We are now in position to solve the existence problem for referee squares. Denote $[a, b]$ as the set of odd integers between $a$ and $b$ and let $R$ denote the set of positive integers $n$ such that there exists a referee square of order $n$.

**Lemma 2.4** If $7 \leq n \leq 187$, then there exists a referee square of order $n$.

**Proof:** If $7 \leq n \leq 51$, the result is obtained by Liaw [5]. When $n = 53, 59, 61, 79$, the result is obtained from Lemma 2.1. For $n = 55, 57, 63, 65, 77$ and 145, the result is obtained by Liaw since $n$ is composite.

From a 5-GDD of type $(2m + 1)^5$ (which exists for all $m \geq 2$ [2]), give weight 3 to every point in the first four groups and weight 1 or 3 to the points in the last group. Since both frames of type $3^5$ and $3^{41}$ exist [3], by Theorem 2.3 there exists a frame of type $(6m + 3)^4(2k + 1)$ when $m \leq k \leq 3m + 1$. Clearly, there exists a referee square of order $6m + 3$ since it is composite. When $2 \leq m \leq 8$, a referee square of order $2k + 1$ exists when $3 \leq k \leq 3m + 1$. Therefore, a referee square of order $n$ can be constructed for all odd $n$ such that $13(2m + 1) \leq n \leq 15(2m + 1)$. Apply this to $m = 2, 3, 4, 5$ to obtain $[67, 75] \cup [91, 105] \cup [117, 135] \cup [143, 165] \subset R$.

In a similar manner from a 9-GDD of type $(2m + 1)^9$ give weight 1 to the first eight groups and 1 or 3 to the last group to obtain a frame of type $(2m + 1)^8(2k + 1)$ for all $m \leq k \leq 3m + 1$. This is possible since there exist frames of type $1^9$ and $1^83^1$ (see [4]). When $3 \leq m \leq 10$, there exists a referee square for each of the possible hole sides in the frame. Hence, we can construct a referee square of order $n$ when $18m + 9 \leq n \leq 22m + 11$. Take $m = 4, 5, 6, 8$ to obtain $[81, 99] \cup [99, 121] \cup [117, 143] \cup [153, 187] \subset R$. 

**Corollary 2.5** There exists a referee square for order $n$ if and only if $n$ odd and $n \geq 7$ or $n = 3$.

**Proof:** Begin with a 5-GDD of type $(2m + 1)^5$ (these exist for all $m > 1$ [2]). Give weight 3 to every point in the first four groups and weight 1 or 3 to the points in last group. Since both frame of type $3^5$ and $3^{41}$ exist ([3]), there exists a frame of type $(6m + 3)^4(2m + 1 + 2i)^1$ for $0 \leq i \leq 2m + 1$. Again, there exists a referee square of order $6m + 3$ since it is composite. Therefore, if
there exists a referee square of order $2m+1+2i$ for $0 \leq i \leq 2m+1$, then there exists a referee square of order $26m+13+2i$ for $0 \leq i \leq 2m+1$. Translating the notation, if $[2m+1, 6m+3] \subset R$, then $[26m+13, 30m+15] \subset R$. Since $[7, 187] \subset R$, by choosing $m \geq 6$ the result follows by induction. \hfill \square

References


