How to Figure a Car Payment

We begin with a formula that seems to have nothing to do with cars or money.

Fact. If $\alpha \neq 1$, then

$$1 + \alpha + \alpha^2 + \alpha^3 + \cdots + \alpha^m = \frac{1 - \alpha^{m+1}}{1 - \alpha}.$$  

Why is it true? Multiply $(1 - \alpha)(1 + \alpha + \alpha^2 + \alpha^3 + \cdots + \alpha^m)$. If you’re real careful, you’ll get:

$$1 + \alpha + \alpha^2 + \alpha^3 + \cdots + \alpha^m + \alpha - \alpha^2 - \alpha^3 - \cdots - \alpha^{m+1}.$$  

Everything cancels except the terms on the ends, leaving us with $1 - \alpha^{m+1}$. That proves the fact (think about it).

(This might remind you of a formula we saw when we proved the derivative power rule—$(x^n)' = nx^{n-1}$—namely:

$$A^n - B^n = (A - B)(A^{n-1} + A^{n-2}B + \cdots + AB^{n-2} + B^{n-1}).$$)  

In fact, it is the same formula, with $A = 1$ and $B = \alpha$.)

Now, let’s suppose we’ve borrowed $10,000 at 8 percent, which we’ll pay off in 36 equal monthly payments. How much will that monthly payment be? Let’s call the payment amount $P$. Our first payment will be made one month from now. To the lender, it has a present value equal to $P$ divided by $1 + (.08/12)$. Another way to phrase that is, the present value of the first monthly payment equals $P$ times $(1 + (.08/12))^{-1}$. The second payment has a present value of $P$ times $(1 + (.08/12))^{-2}$. The third payment’s present value is $P(1 + (.08/12))^{-3}$; and so on. We’re going to be using this number $(1 + (.08/12))^{-1}$ so much that we should give it a name. Let’s set

$$\alpha = (1 + (.08/12))^{-1}. \quad (1)$$

The choice of the symbol $\alpha$ is not an accident.

The present value of all of the monthly payments is

$$P\alpha + P\alpha^2 + P\alpha^3 + \cdots + P\alpha^{36} = P(\alpha + \alpha^2 + \cdots + \alpha^{36})$$

$$= P\alpha(1 + \alpha + \alpha^2 + \cdots + \alpha^{35}). \quad (2)$$

Note the tricky change in the second line. By our formula,

$$1 + \alpha + \alpha^2 + \cdots + \alpha^{35} = \frac{1 - \alpha^{36}}{1 - \alpha}.$$  

Therefore, the present value of all of the monthly payments must be (substitute this into (2)):

$$P\alpha \left( \frac{1 - \alpha^{36}}{1 - \alpha} \right).$$
This must equal the value of the loan, which is $10,000. In other words, we must have

\[ 10,000 = P\alpha \left(1 - \alpha^{36}\right), \]

or

\[ P = \frac{10,000}{\alpha} \left(\frac{1 - \alpha}{1 - \alpha^{36}}\right). \quad (3) \]

For our problem, \( \alpha = .993377 \) and \( \alpha^{36} = .787255 \). When we plug these into (3) we get \( P = 313.364 \), which rounds to $313.36.

Suppose the interest rate is 9%, the loan is for $8500, and we’re borrowing for 24 months. Do we have to do the computation all over again? No: not if we work out a general formula. The crucial numbers are \( L \) (the amount of the loan), \( r \) (the interest rate), and \( N \) (the number of monthly payments). The most significant change will be in the number we’ve called \( \alpha \). For an interest rate \( r \), the value of \( \alpha \) we get will be \( (1+r/12)^{-1} \). (Compare this to (1), where \( r = .08 \).) But also, where we have a ‘36’ in (3), we’ll have an \( N \) in the general formula. Putting it all together, we get

\[ P = \frac{L}{\alpha} \left(\frac{1 - \alpha}{1 - \alpha^{N}}\right), \quad (4) \]

where \( \alpha = (1 + r/12)^{-1} \).

We can make this a little simpler—with a little algebra. Notice that \( 1/\alpha = 1 + r/12 \) and that

\[ 1 - \alpha = 1 - \frac{1}{1 + r/12} = \frac{r/12}{1 + r/12}. \]

If we plug these into (4) we get

\[ P = L(1 + r/12) \left(\frac{r/12}{(1 + r/12)(1 - \alpha^{N})}\right) = \frac{Lr}{12(1 - \alpha^{N})}. \quad (5) \]

If \( r = .09 \), \( L = 8500 \), and \( N = 24 \), then \( \alpha = .992556 \), and the monthly payment is $388.32.

Another example: mortgage. You buy a house for $250,000, taking out a 30-year mortgage at 7% interest. What is the monthly payment? Here \( L = 250,000 \), \( r = .07 \), and \( N = 360 \). We compute \( \alpha \) and get .9942. Plugging these into the formula (5), we get a monthly payment of $1663.26.

A final wrinkle: non-monthly payments. Some lenders allow you to make quarterly, semi-annual, or twice-monthly payments. How can we adapt our formula (5) to handle these? What’s changed here is the number of times a year we’ll be compounding the
interest. It was 12, and now it’s \( n \). It looks like all we have to do is replace the ‘12’ with an \( n \)—being careful to also replace the ‘hidden’ 12 in the formula for \( \alpha \) with an \( n \). But is this all? Not quite. The number \( N \) was the number of months in the loan. Now it’s the number of \textit{payments}. If you look back to our original derivation, you’ll see that that’s what it always was, but our emphasis on monthly payments tended to disguise this fact.

The new, final, super-duper general formula for loan amortization is

\[
P = \frac{Lr}{n(1 - \alpha^N)},
\]

where \( L \) is the loan amount, \( r \) is the (annual) interest rate, \( n \) is the number of times per year that interest will be compounded, \( N \) is the number of payments, and

\[
\alpha = (1 + (r/n))^{-1}.
\]

\textbf{Example: a mortgage with quarterly payments.} We’ll use the same loan amount as before—$250,000—with the same 7\% interest rate, and also over 30 years. Now, with \( n = 4 \), our value of \( \alpha \) is \((1 + .07/4)^{-1} = .982801\). The number \( N \) is 120. Plugging these in, the quarterly payment is $4998.29.

\textbf{Example: a mortgage with twice-monthly payments.} We use the same interest rate, term, and loan amount from the previous problem. Now \( n = 24 \), yielding \( \alpha = (1 + .07/24)^{-1} = .997092\). The number of payments is 720. Plugging it all in, we get a monthly payment of $831.27.

\textbf{A question.} Although the quarterly payment is made every three months, it is actually a little larger than three of the monthly payments ($1663.26) calculated above. Also, the twice-monthly payment is a little smaller than half of the monthly payment. Why do you think this is so?

This is non-trivial mathematics, and I know some of you are finding it tough. But this math has a payoff: it gives you \textit{power}. 