Math 242 Homework Set #1  
Spring, 2006

Assigned January 26. Due February 7. The problems have equal credit.

1. Let $f : (0, 1) \mapsto \mathbb{R}$ and suppose that $f'(x)$ exists for all $x \in (0, 1)$. Suppose furthermore that there is a number $M$ such that $|f'(x)| \leq M$ for all $x \in (0, 1)$. Show that $\lim_{x \to 0^+} f(x)$ and $\lim_{x \to 1^-} f(x)$ both exist as real numbers. Give an example to show that these limits can fail to exist—even as infinite quantities—if $f'$ is not bounded by some $M$.

2. Let $f : (0, 1) \mapsto \mathbb{R}$ be continuous, and suppose that $f'(x)$ exists (finite or infinite) for all $x \in (0, 1)$ with the possible exception of some $c \in (0, 1)$. Show that if the limit

$$\lim_{x \to c} f'(x)$$

exists and equals $A$, a real number, then $f'(c)$ also exists and equals $A$.

3. Let $f : \mathbb{R} \mapsto \mathbb{R}$ have a finite derivative at all points, and suppose that

$$\lim_{x \to \infty} f'(x) = M,$$

a real number. Show that the limit

$$\lim_{x \to \infty} f(x)/x$$

exists and equals $M$. (Hint: begin by proving the result under the assumption that $M = 0$.)

4. Suppose $f : [0, 1] \mapsto \mathbb{R}$ is continuous, $f(0) = 0$, and $f'(x)$ is finite for all $x \in (0, 1)$. Suppose furthermore that $f'(x)$ is an increasing function on $(0, 1)$. Show that $f(x)/x$ is also increasing on $(0, 1)$. Give an example to show that this conclusion can fail if $f(0) \neq 0$.

5. Let $f : (0, 1) \mapsto \mathbb{R}$ be non-negative and have a finite third derivative $f'''$ everywhere on $(0, 1)$. Suppose that $f(x_1) = f(x_2) = 0$ for two distinct points $x_1$ and $x_2$ in $(0, 1)$. Show that there is a $c \in (0, 1)$ where $f'''(c) = 0$. Give an example to show that this conclusion can fail if $f$ assumes negative values on $(0, 1)$.