1. A function \( f : [a, b] \mapsto \mathbb{R} \) is said to be \textit{absolutely continuous} on \([a, b]\) if, for every \( \epsilon > 0 \), there is \( \delta > 0 \) such that, whenever \((a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)\) are disjoint intervals in \([a, b]\) satisfying

\[
\sum_{j=1}^{n} (b_j - a_j) < \delta,
\]

we also have

\[
\sum_{j=1}^{n} |f(b_j) - f(a_j)| < \epsilon.
\]

Show: a) if \( f \) is absolutely continuous on \([a, b]\), then it is uniformly continuous on \([a, b]\);

b) if \( f \) is absolutely continuous on \([a, b]\), it is of bounded variation on \([a, b]\).

\textit{A solution.} WLOG, \([a, b] = [0, 1]\).

a) Let \( \epsilon > 0 \) and find the appropriate \( \delta > 0 \). If \( x \) and \( y \) lie in \([0, 1]\) and \( 0 < y - x < \delta \), then \( |f(x) - f(y)| < \epsilon \), which is uniform continuity.

b) Pick \( \epsilon = 1 \) and the appropriate \( \delta > 0 \). Let \( N > \delta^{-1} \) be a positive integer and let \( P_N \) be the partition \( P_N = \{0 < 1/N < 2/N < 3/N < \cdots < (N - 1)/N < 1\} \). Now let \( P = \{0 = x_0 < x_1 < x_2 < \cdots < x_n = 1\} \) be any partition. Without loss of generality, we may assume that \( P_N \subset P \). Group the integers from 1 to \( n \) into \( N \) disjoint families \( F_1, \ldots, F_N \), where \( F_j \) consists of all the numbers \( k \) (\( 1 \leq k \leq n \)) such that \([x_{k-1}, x_k] \subset [(j - 1)/N, j/N]\). (This kind of grouping makes sense because \( P_N \subset P \).) We can thus write:

\[
\sum_{P} |\Delta f_k| = \sum_{j=1}^{N} \sum_{k: k \in F_j} |\Delta f_k|.
\]

But

\[
\sum_{k: k \in F_j} (x_k - x_{k-1}) = 1/N < \delta,
\]

and therefore

\[
\sum_{k: k \in F_j} |\Delta f_k| < \epsilon = 1
\]

for each \( j \). Summing on \( j \) from 1 to \( N \) yields

\[
\sum_{P} |\Delta f_k| = \sum_{j=1}^{N} \sum_{k: k \in F_j} |\Delta f_k| \leq \sum_{j=1}^{N} 1 = N,
\]

implying \( V_f[0, 1] \leq N \). QED.

2. Let \( S = \{q_1, q_2, \ldots\} \) be the set of rational numbers in \([0, 1]\), and define

\[
f(x) = \begin{cases} 
1/k & \text{if } x = q_k; \\
0 & \text{if } x \notin S.
\end{cases}
\]
Show that $f$ is Riemann integrable on $[0, 1]$, and find $\int_0^1 f \, dx$.

A solution. The value of the integral is 0. To prove this, it’s enough to show that, for every $\epsilon > 0$, there is a partition $P$ of $[0, 1]$ such that $U(P, f) < \epsilon$. Let $1/N < \epsilon/2$, where $N$ is a positive integer. We may assume that $N$ is so large that 0 and 1 show up in the list $q_1, q_2, \ldots, q_N$. Let $\{[a_j, b_j]\}_{j=1}^N$ be a collection of disjoint subintervals of $[0, 1]$ such that each $q_j \neq 0$ or 1 lies in $(a_j, b_j)$, 0 lies in an interval of the form $[0, b_j)$, 1 lies in an interval of the form $(a_j, 1]$, and $\sum_{j=1}^N (b_j - a_j) < \epsilon/2$. Let $P$ the partition consisting of the points $\{a_1, b_1, a_2, \ldots, b_N\}$. (This set will automatically contain 0 and 1.) This partition gives rise to two types of subintervals: a) intervals of the form $[a_j, b_j]$, on which $f$ is no bigger than 1, and the sum of whose lengths is $< \epsilon/2$; b) intervals on which $f$ is no bigger than $1/N < \epsilon/2$, and the sum of whose lengths is no more than 1. Summing $M_k(f)\Delta x_k$ over the first group adds up to $< \epsilon/2$ ($f$ might be “big”, but the intervals’ lengths sum to a small number), while summing $M_k(f)\Delta x_k$ over the second is also less than $\epsilon/2$ (small $f$ over intervals whose lengths sum to no more than 1). Combining them, we get $U(P, f) < \epsilon$.

QED.

3. A function $\phi : [a, b] \to \mathbb{R}$ is called a step function if there is a partition $P_\phi$ on $[a, b]$ such that $\phi$ is constant on each open subinterval $(x_{k-1}, x_k)$ of $P_\phi$. Suppose that $\phi$ is a step function with associated partition $P_\phi$, with values $\phi(x) = \begin{cases} \alpha_k & \text{if } x \in (x_{k-1}, x_k); \\ \beta_k & \text{if } x = x_k \in P_\phi = \{a = x_0 < \cdots < x_n = b\}. \end{cases}$

Show that $\phi$ is Riemann integrable, and that

$$\int_a^b \phi(x) \, dx = \sum_{k=1}^n \alpha_k(x_k - x_{k-1}).$$

A solution. We will say that $\phi$ is a point function if there is a $p \in [a, b]$ such that

$$\phi(x) = \begin{cases} 1 & \text{if } x = p; \\ 0 & \text{otherwise}; \end{cases}$$

and we will say that $\phi$ is an interval function if there is an open interval $I = (\alpha, \beta) \subset [a, b]$ such that

$$\phi(x) = \begin{cases} 1 & \text{if } x \in I; \\ 0 & \text{otherwise.} \end{cases}$$

We will show that the integral of every point function is 0 and the integral of every interval function is $\beta - \alpha$. The result for general step functions will then follow by taking linear combinations. For the point function there is no problem. If $P$ is any partition such that $\|P\| < \epsilon$, then $U(P, \phi) < 2\epsilon$, because the Riemann sum has at most two non-zero terms, and both are smaller than $\epsilon$. Since all of $\phi$’s lower sums are zero, that finishes the proof for
point functions. For interval functions, I will assume that $a < \alpha < \beta < b$ (the cases where we might have equality at the endpoints are similar to this one). Let $P$ be the partition

$$P = \{a < \alpha - \epsilon < \alpha + \epsilon < \beta - \epsilon < \beta + \epsilon < b\},$$

where we may assume that $\epsilon$ is small enough to make all of those inequalities true. When we evaluate $U(P, \phi)$ we get $\beta - \alpha + 2\epsilon$, and when we evaluate $L(P, \phi)$ we get $\beta - \alpha - 2\epsilon$. Thus, for every $\epsilon$, there is a partition $P$ such that

$$\beta - \alpha - 2\epsilon = L(P, \phi) < U(P, \phi) = \beta - \alpha + 2\epsilon.$$

This implies $\phi \in R$ and it gives the value of the integral, because it shows that $\beta - \alpha \leq \sup_P L(P, \phi) \leq \inf_P U(P, \phi) \leq \beta - \alpha$. QED.

4. Use #3 to prove the following: If $[a, b]$ is a closed bounded interval and $\{(a_j, b_j)\}^\infty_1$ is any collection of open intervals such that

$$[a, b] \subset \bigcup^\infty_1 (a_j, b_j),$$

then

$$b - a \leq \sum^\infty_1 (b_j - a_j).$$

(Hint: begin by reducing to a finite subcover.)

A solution. We may assume that $[a, b] = [0, 1]$ and that every $(a_j, b_j) \subset [-1, 2]$. Following the hint, we have

$$[0, 1] \subset \bigcup^n_1 (a_j, b_j),$$

(1) after taking a finite subcover and renumbering. Set

$$f(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1; \\ 0 & \text{otherwise;} \end{cases}$$

$$g_j(x) = \begin{cases} 1 & \text{if } a_j < x < b_j; \\ 0 & \text{otherwise.} \end{cases}$$

Then (1) implies that $f(x) \leq \sum^n_1 g_j(x)$ on $[-1, 2]$. Now, by exercise #3,

$$1 = \int_{-1}^{2} f(x) \, dx \leq \int_{-1}^{2} \left( \sum^n_1 g_j(x) \right) \, dx = \sum^n_1 (b_j - a_j) \leq \sum^\infty_1 (b_j - a_j).$$

QED.

5. Define $\alpha : [-1, 1] \mapsto \mathbb{R}$ by

$$\alpha(x) = \begin{cases} 1 & \text{if } x = 0; \\ 0 & \text{otherwise.} \end{cases}$$
Show that, if \( f : [-1, 1] \mapsto \mathbb{R} \) is continuous, then \( f \in R(\alpha) \) on \([-1, 1]\), and find a formula for \( \int_{-1}^{1} f \, d\alpha \).

A solution. Let \( \epsilon > 0 \), and let \( 0 < \delta < 1 \) be such that \( |x| < \delta \) implies \( |f(x) - f(0)| < \epsilon \). Let \( P_{\epsilon} \) be any partition on \([-1, 1]\) containing the points 0, \(-\delta/2\), and \(\delta/2\). Let \( P \) be finer than \( P_{\epsilon} \). Let \( z_1 < 0 < z_2 \) be the 3 closest points (including 0) in \( P \) that are closest to 0. Then each \( |z_i| \leq \delta/2 < \delta \). Any Riemann-Stieltjes sum \( S(P, f, \alpha) \) will be equal to \( f(t_1) - f(t_2) \), where \( z_1 \leq t_1 \leq 0 \leq t_2 \leq z_2 \). But then we also have \( |t_i| < \delta \). Therefore:

\[
|f(t_1) - f(t_2)| \leq |f(t_1) - f(0)| + |f(0) - f(t_2)| < 2\epsilon,
\]

showing that \( f \in R(\alpha) \) and that the integral equals 0. (Note: the proof only requires \( f \) to be continuous at 0.)