1. Let \( f : [a, b] \to \mathbb{R} \) be continuous and let \( M = \sup \{|f(x)| : x \in [a, b] \} \). Show that

\[
\lim_{n \to \infty} \left( \int_a^b |f(x)|^n \, dx \right)^{1/n}
\]

equals \( M \).

See next problem.

2. As in #1, let \( f : [a, b] \to \mathbb{R} \) be continuous and let \( M = \sup \{|f(x)| : x \in [a, b] \} \). Suppose that \( \alpha : [a, b] \to \mathbb{R} \) is strictly increasing. Show that

\[
\lim_{n \to \infty} \left( \int_a^b |f(x)|^n \, d\alpha(x) \right)^{1/n}
\]

equals \( M \).

A solution. If \( M = 0 \) there is nothing to prove. So, assume \( M > 0 \). For all \( n \),

\[
\int_a^b |f(x)|^n \, d\alpha(x) \leq M^n (\alpha(b) - \alpha(a)),
\]

therefore, as \( n \to \infty \),

\[
\limsup_{n \to \infty} \left( \int_a^b |f(x)|^n \, d\alpha(x) \right)^{1/n} \leq \limsup_{n \to \infty} M^{1/n} (\alpha(b) - \alpha(a))^{1/n} = M,
\]

because \( \alpha(b) - \alpha(a) > 0 \). Now let \( 0 < \epsilon < M \). There is an \( x_0 \in [a, b] \) such that \( |f(x_0)| = M \), and there is is an interval \([c, d]\) containing \( x_0 \), with \( d - c > 0 \), such that \( |f(x)| \geq M - \epsilon \) on all of \([c, d]\). Thus:

\[
\liminf_{n \to \infty} \left( \int_a^b |f(x)|^n \, d\alpha(x) \right)^{1/n} \geq \liminf_{n \to \infty} \left( \int_c^d |f(x)|^n \, d\alpha(x) \right)^{1/n} \geq \liminf_{n \to \infty} (M - \epsilon)^{1/n} (\alpha(d) - \alpha(c))^{1/n} = \lim_{n \to \infty} (M - \epsilon)^{1/n} (\alpha(d) - \alpha(c))^{1/n} = M - \epsilon,
\]

where the last equation follows because \( \alpha(d) - \alpha(c) > 0 \). This holds for all small \( \epsilon > 0 \), and that proves the result.
3. Let \( f \) be Riemann integrable on \([0, 1]\) and set \( A = \int_0^1 f \, dx \). Show that, for every \( \epsilon > 0 \), there is a \( \delta > 0 \) such that, if \( P \) is any partition on \([0, 1]\) and \( \|P\| < \delta \), then \( |S(P, f) - A| < \epsilon \). In other words, show that, if \( f \) is Riemann integrable, then

\[
\int_0^1 f(x) \, dx = \lim_{\|P\| \to 0} S(P, f),
\]

independent of the choice of the points \( t_k \) in the Riemann sums. This is one of Apostol’s exercises, and he gives a good hint for it.

Apostol’s hint basically gives away the store, but you have to know how to justify the inequalities.

4. Let \( f : [a, b] \to \mathbb{R} \) be continuous and suppose that \( \alpha : [a, b] \to \mathbb{R} \) is of bounded variation. Let \( V = V_\alpha \), \( \alpha \)'s total variation function; i.e., for any \( x \in [a, b] \), \( V(x) = \) the total variation of \( \alpha \) on \([a, x]\). From class we know that \( f \) and \(|f|\) both belong to \( R(\alpha) \cap R(V) \) on \([a, b]\). Show that

\[
|\int_a^b f \, d\alpha| \leq \int_a^b |f| \, dV.
\]

A solution. We know that both integrals exist, because \( f \) is continuous. Let \( P = \{a = x_0 < \cdots < x_n = b\} \) be any partition of \([a, b]\). Then:

\[
\left| \sum_{k=1}^n f(t_k) \Delta \alpha_k \right| \leq \sum_{k=1}^n |f(t_k)| \|\Delta \alpha_k\| \\
\leq \sum_{k=1}^n |f(t_k)| \Delta V_k.
\]

But, for any \( \epsilon > 0 \), we can choose \( P \) so that the first quantity is within \( \epsilon \) of

\[
|\int_a^b f \, d\alpha|
\]

and the last quantity is within \( \epsilon \) of

\[
\int_a^b |f| \, dV.
\]

That does it.

5. Let \( f : [a, b] \to \mathbb{R} \) be bounded. Suppose that \( \alpha : [a, b] \to \mathbb{R} \) is continuous and that \( \alpha' \) is finite and bounded everywhere in \((a, b)\). Show that, if the two integrals

\[
\int_a^b f(x) \, d\alpha(x)
\]
and
\[ \int_a^b f(x) \alpha'(x) \, dx \]
both exist, then they must be equal. Warning: this is not quite what we proved in class!

A solution. Let \( \epsilon > 0 \). We can find a partition \( P \) such that
\[
\left| S(P, f, \alpha) - \int_a^b f(x) \, d\alpha(x) \right| < \epsilon
\]
and
\[
\left| S(P, f\alpha') - \int_a^b f(x) \alpha'(x) \, dx \right| < \epsilon
\]
for all choices of the intermediate points \( t_k \). However, using the MVT, we can choose the points \( t_k \) so that \( S(P, f\alpha') = S(P, f, \alpha) \). That proves it.