Math 331 Homework #2 Solutions Spring, 2010

Assigned February 10. Due February 19. The five numbered problems have equal credit.

1. Find, with justification, the radii of convergence of the following power series. In each case, assume that \( \alpha \) is a non-zero complex number. Your answers might depend on \( \alpha \).

   a) \[ \sum_{n=1}^{\infty} \alpha^n z^n. \]
   b) \[ \sum_{n=1}^{\infty} \alpha^n z^{2n}. \]
   c) \[ \sum_{n=1}^{\infty} \alpha^{2n} z^{n!}. \]

   **Solutions.** a) Apply the Root Test: \( |\alpha^n|^{1/n} = |\alpha|^n \), which goes to infinity if \( |\alpha| > 1 \), is 0 if \( |\alpha| < 1 \), and 1 if \( |\alpha| = 1 \). The respective radii of convergence are 0, infinity, and 1. b) Apply the Root Test again: \( |\alpha^n|^{1/n^2} = |\alpha|^{-1/n} \rightarrow 1 \), since \( \alpha \neq 0 \). The radius of convergence is 1. c) Since \( 2^n/n! \rightarrow 0 \), the same argument as before shows that the radius of convergence is 1.

2. Suppose that the power series \( \sum_{n=0}^{\infty} a_n z^n \) has radius of convergence equal to 3, and \( k \) is a positive integer. Find the radii of convergence of \( \sum_{n=0}^{\infty} a_n^{k} z^n \) and \( \sum_{n=0}^{\infty} a_n z^{kn} \). (Your answers might depend on \( k \).) Now suppose \( \sum_{n=0}^{\infty} a_n z^n \) has radius of convergence equal to \( 1/3 \). What are the radii of convergence of \( \sum_{n=0}^{\infty} a_n^{k} z^n \) and \( \sum_{n=0}^{\infty} a_n z^{kn} \)?

   **Solutions.** We use the elementary fact that if \( \{\alpha_n\} \) is any sequence of non-negative numbers and \( \beta > 0 \) then 
   \[
   \limsup(\alpha_n^\beta) = (\limsup \alpha_n)^\beta.
   \]
   The respective radii of convergence are \( 3^k, 3^{1/k}, 3^{-k}, \) and \( 3^{-1/k} \).

3. Suppose that \( K \subset \mathbb{C} \) is compact and non-empty, and define \( C(K) \) to be the family of continuous functions \( f : K \rightarrow \mathbb{C} \) (where “continuous” means “continuous with respect to the usual, absolute-value metric on \( \mathbb{C} \)”). For \( f \in C(K) \) define 
   \[
   \|f\|_K \equiv \sup_{z \in K} |f(z)|.
   \]
   Show that \( (C(K), \| \cdot \|_K) \) is a normed vector space. Show that \( C(K) \) is complete with respect to the metric \( d_K(\cdot, \cdot) \) defined by 
   \[
   d_K(f, g) \equiv \|f - g\|_K.
   \]

   **Solution.** If \( f \) and \( g \) are continuous on \( K \) and \( \alpha \) and \( \beta \) are complex numbers then \( \alpha f + \beta g \) is continuous; hence \( C(K) \) is a vector space. The norm \( \| \cdot \|_K \) is defined for all \( f \in C(K) \) because \( f[K] \) is compact, hence bounded in \( \mathbb{C} \). It’s clearly non-negative, and it equals 0 if and only if \( f(z) = 0 \) for all \( z \in K \). If \( \alpha = 0 \) then \( \sup_{z \in K} |\alpha f(z)| = 0 = 0 \cdot \|f\|_K \); and if \( \alpha \neq 0 \) then \( \sup_{z \in K} |\alpha f(z)| = |\alpha| \sup_{z \in K} |f(z)| \), by elementary (Math 241) properties of the supremum. If \( f \) and \( g \) belong to \( C(K) \) then, for all \( z \in K \),
   \[
   |f(z) + g(z)| \leq |f(z)| + |g(z)| \leq \|f\|_K + \|g\|_K.
   \]
implying \( \| \cdot \|_K \) satisfies the triangle inequality. Therefore \((C(K), \| \cdot \|_K)\) is a normed vector space. Let \( \{f_n\} \subset C(K) \) be Cauchy. Then for every \( \epsilon > 0 \) there is an \( N \) such that if \( m \) and \( n \) are \( \geq N \) and \( z \in K \),

\[
|f_m(z) - f_n(z)| \leq \|f_m - f_n\|_K < \epsilon;
\]
i.e., \( \{f_n\} \) is uniformly Cauchy on \( K \). There exists a function \( f : K \mapsto \mathbb{C} \) such that \( f_n \to f \) uniformly. Since each \( f_n \) is continuous, so is \( f \), and \( f \in C(K) \). Let \( \epsilon > 0 \). Since \( f_n \to f \) uniformly, there is an \( N \) such that, if \( n \geq N \) and \( z \in K \),

\[
|f(z) - f_n(z)| < \epsilon.
\]

Taking the supremum over \( z \in K \), we get \( \|f - f_n\|_K \leq \epsilon \) for \( n \geq N \). Therefore \( f_n \to f \) in the \( d_K(\cdot, \cdot) \) metric, and \( C(K) \) is complete.

4. Let \( \{K_n\}_{1}^{\infty} \) be a sequence of compact subsets of \( \mathbb{C} \), and suppose the sequence is nested, meaning that \( K_{n+1} \subset K_n \) for all \( n \). Suppose that \( \Omega \) is an open subset of \( \mathbb{C} \) such that \( \cap_{1}^{\infty} K_n \subset \Omega \). Show that there exists a finite \( N \) such that \( K_N \subset \Omega \). Show that this conclusion can fail if the \( K_n \)’s are merely assumed to be closed.

\textit{Solutions.} This is a disguised form of Proposition 4.4 on page 21; we will prove it using sequences. Define \( E_n = K_n \setminus \Omega \). The sequence \( \{E_n\}_{1}^{\infty} \) is also a nested sequence of compact sets and

\[
\cap_{1}^{\infty} E_n = (\cap_{1}^{\infty} K_n) \setminus \Omega = \emptyset, \tag{*}
\]

because \( \cap_{1}^{\infty} K_n \subset \Omega \). Suppose no \( E_n = \emptyset \). For each \( n \), pick \( z_n \in E_n \subset E_1 \). Let \( \{z_{n_k}\} \subset E_1 \) be a convergent subsequence such that \( z_{n_k} \to p \in E_1 \). Because the \( E_n \)’s are nested, this \( p \) will have to belong every \( E_n \) as well. Therefore \( p \in \cap_{1}^{\infty} E_n = \emptyset \), a contradiction. Therefore some \( E_N \) is empty, implying some \( K_N \subset \Omega \). For the counterexample, let \( \Omega = \emptyset \) and \( K_N = \{z \in \mathbb{C} : |z| \geq N\} \).

5. Let \( \{K_n\}_{1}^{\infty} \) be a nested sequence of connected, compact subsets of \( \mathbb{C} \). Show that \( \cap_{1}^{\infty} K_n \) is connected. (Problem \#4 might come in useful here.) Give an example of a nested sequence \( \{F_n\}_{1}^{\infty} \) of connected closed subsets of \( \mathbb{C} \) such that \( \cap_{1}^{\infty} F_n \) is not connected.

\textit{Solutions.} Set \( E = \cap_{1}^{\infty} K_n \) and suppose, by contradiction, that \( (A, B) \) is disconnection for \( E \) such that \( A \cap B = \emptyset \). Define \( \Omega = A \cup B \). By Problem \#4, we must have \( K_N \subset \Omega \) for some \( N \). But then

\[
\emptyset \neq E \cap A \subset K_N \cap A \\
\emptyset \neq E \cap B \subset K_N \cap B,
\]
implying that \( (A, B) \) is also a disconnection for \( K_N \), which is a contradiction. For the counterexample, let \( K_n \) be \( \{x + iy : x = 1\} \cup \{x + iy : x = -1\} \cup \{x + iy : y \geq n\} \). You’ll understand this better if you sketch the sets.