Assigned October 19. Due November 3. The problems are worth 14 points each, with 2 points added to make 100.

In the exercises, $\mathcal{L}$ denotes the $\sigma$-algebra of Lebesgue measurable sets on $\mathbb{R}$, $m$ denotes Lebesgue measure, and $m^*$ denotes Lebesgue outer measure.

1. Let $E \subset [0, 1)$ be the Lebesgue nonmeasurable set we “constructed” (via the Axiom of Choice!) in class. Show that if $F \subset E$ and $F \in \mathcal{L}$, then $m(F) = 0$.

2. Show that if $H \subset \mathbb{R}$ and $m^*(H) > 0$, then $H$ contains a Lebesgue nonmeasurable subset. (Hint: First show that it’s enough to prove this for $H \subset [0, 1)$.)

3. Give an example of a function $f : \mathbb{R} \mapsto \mathbb{R}$ such that $f^{-1}\{x\} \in \mathcal{L}$ for all $x \in \mathbb{R}$, but $f$ is NOT Lebesgue measurable.

4. Give an example of an uncountable family of Lebesgue measurable functions $\{f_\alpha\}_\alpha$ (where the $\alpha$’s run over some index set), mapping from $\mathbb{R}$ to $\mathbb{R}$, such that: a) $|f_\alpha(x)| \leq 1$ for all $\alpha$ and all $x$; b) $g(x) \equiv \sup_\alpha f_\alpha(x)$ is NOT Lebesgue measurable.

5. Show that if $\{f_\alpha\}_\alpha$ is an arbitrary (possibly uncountable) family of continuous functions mapping from $\mathbb{R}$ to $\mathbb{R}$, then $g(x) \equiv \sup_\alpha f_\alpha(x)$, mapping from $\mathbb{R}$ to $\mathbb{R}$, is Lebesgue measurable.

6. Let $(X, \mathcal{M})$ be a measurable space, and let $f_n : X \mapsto \overline{\mathbb{R}}$ be an arbitrary sequence of measurable functions. Let $E = \{x \in X : \lim_n f_n(x) \text{ exists}\}$, where the limit can be either finite or infinite. Show that $E \in \mathcal{M}$. Show that if we define:

$$f(x) \equiv \begin{cases} \lim_n f_n(x) & \text{if } x \in E; \\ 0 & \text{if } x \notin E; \end{cases}$$

then $f$ is measurable.

7. For sets $A$ and $B$, we define $A \Delta B \equiv (A \setminus B) \cup (B \setminus A)$. Show that, if $E \in \mathcal{L}$ and $m(E) < \infty$, then, for every $\epsilon > 0$, there is a set $U$, equal to a finite union of bounded open intervals, such that $m(E \Delta U) < \epsilon$. 