Diophantine Analysis and Brownie Pan Optimization
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Abstract.
We show a surprising and fun connection between a classic calculus problem and a method due to Diophantus.

A familiar calculus exercise goes like this. We are given a rectangular sheet of thin metal. We have to cut out identical squares from the sheet’s corners, fold up the resulting flaps, and (after some welding, one presumes) make a brownie pan. The question is, how big should the squares be to maximize the pan’s volume?

Let’s first suppose the sheet to be a square, measuring $L \times L$. If we cut out corner squares measuring $t \times t$, where $0 \leq t \leq L/2$, the brownie pan will have dimensions $(L - 2t) \times (L - 2t) \times t$, and volume equal to $f(t) \equiv t(L - 2t)^2$. Taking the derivative, we see

$$f'(t) = 12t^2 - 8Lt + L^2 = (L - 6t)(L - 2t).$$

Since $f(L/2) = 0$, the relevant critical number is at $t = L/6$. That’s the optimal size for the cut-out squares, and the maximum volume is $f(L/6) = 2L^3/27$.

The “rectangular” problem would seem to be a natural generalization of this. Let the sheet measure $A \times B$, with $0 < A < B$. If we cut out corner squares measuring $t \times t$ (with $0 \leq t \leq A/2$) and fold up the sides, the brownie pan will have volume equal to $g(t) \equiv t(A - 2t)(B - 2t) = 4t^3 - 2(A + B)t^2 + ABt$. The derivative of this is $g'(t) = 12t^2 - 4(A + B)t + AB$. Just set it equal to 0 and solve for $t$! What can possibly go wrong?

If the reader tries this, he will quickly learn why elementary calculus texts never state the rectangular brownie pan problem in such a general form. The square sheet might measure $L \times L$, but, in every textbook I’ve seen, the rectangular sheet has dimensions of $3 \times 8$, $7 \times 15$, or some other carefully chosen pair of integers. These dimensions make $g'(t)$ a factorable polynomial, so that the calculus problem can be solved on a blackboard. With dimensions of $3 \times 7$, $7 \times 16$, or—worst of all—$A \times B$, you get a mess: I speak from experience.

If the dimensions $A$ and $B$ are integers, the derivative polynomial, $12t^2 - 4(A + B)t + AB$, is factorable if and only if its discriminant is a perfect square. The discriminant is $16(A^2 - AB + B^2)$. Since $16 = 4^2$, the discriminant will be a perfect square if and only if $A^2 - AB + B^2$ is one too. The reader can readily check that the pairs $(3, 8)$ and $(7, 15)$ have this property.

It is not clear how one goes about finding such pairs.

Here is where Diophantus can help. Diophantus might have phrased our problem this way: Given a (rational) number $C$, find (rational) numbers $A$ and $B$ so that the square of $A$ plus the square of $B$, minus the product of $A$ and $B$, is equal to the square of $C$. To solve this problem, he might well have used the method he applied elsewhere: Given a (rational) number $C$, find (rational) numbers $A$ and $B$ such that the square of $A$ plus the
square $B$ is equal to the square of $C$. In other words, given a rational $C$, find rational $A$ and $B$ such that
\[ A^2 + B^2 = C^2. \]  
(1)

This equation is, of course, very famous. Fermat’s Last Theorem states that (1) has no nontrivial, rational solutions for integral exponents larger than 2. It was in the margins of his own copy of Diophantus’ *Arithmetica*—and at the discussion of just this problem—that Fermat claimed to have found a proof of this fact, but that the proof was too long to fit in the margin.

We won’t explore (1). Instead, we will apply Diophantus’ elegant method to find all the integral solutions of
\[ A^2 - AB + B^2 = C^2. \]  
(2)

After dividing both sides of (2) by $C^2$, we see that it is sufficient to find all the rational solutions of
\[ A^2 - AB + B^2 = 1, \]  
(2′)

and that is what we shall do.

If (2′) has a rational solution $(A, B)$, then there is a rational $m$ such that $B = 1 - mA$. When we substitute this into (2′) and expand, we get
\[ (m^2 + m + 1)A^2 - (2m + 1)A + 1 = 1. \]

Subtracting 1 from both sides leaves
\[ (m^2 + m + 1)A^2 - (2m + 1)A = 0, \]

which Diophantus wouldn’t have written. But, like us, he would have concluded:
\[ (m^2 + m + 1)A^2 = (2m + 1)A. \]

After dividing by $A$ (if $A = 0$ the solution is trivial), and dividing both sides by $m^2 + m + 1$, he would get
\[ A = \frac{2m + 1}{m^2 + m + 1}. \]

The formula $B = 1 - mA$ then yields
\[ B = \frac{1 - m^2}{m^2 + m + 1}. \]

Diophantus would only have allowed positive values of $m$. If we insist on getting only positive $A$’s and $B$’s, then we require that $-1/2 < m < 1$. Plugging in $m = 1/2$, we get
\[ A = \frac{8}{7}, \quad B = \frac{3}{7}, \]
which yields the integral pair \((3, 8)\). If \(m = 2/5\), we get

\[
A = 15/13 \\
B = 7/13,
\]

and we recover \((7, 15)\). Just for fun, let’s put \(m = -11/24\). Turning the crank produces

\[
A = 48/433 \\
B = 455/433,
\]

from which we can conclude that the \(48 \times 455\) brownie pan problem is, in principle, “blackboard solvable”!

**Exercises.**

1. Solve the brownie pan optimization problems for the dimensions \(3 \times 8\) and \(7 \times 15\).

2. Let \(A\) and \(B\) be integers, with \(0 < A < B\). Set \(A' = B - A\). Show that, if \(A^2 - AB + B^2\) is a perfect square, so is \((A')^2 - A'B + B^2\).

3. Let \(A\) and \(B\) be positive numbers such that \(A < B\) and \(A^2 - AB + B^2 = 1\). Show that \(A < 1 < B\).

4. Use the method described above to find all rational solutions of \(A^2 + B^2 = 1\).