Mathematical Games
by M*tr*r*n G*rdn*r

Given a number \( N \), how many numbers are less than it? The solution to this problem could have profound consequences in mathematics and physics.

O Great Ra,
Count me, for I am a number,
Subtract me, for I am a sum of numbers!
O Great Ra,
Thou who countest the numbers!

—"The Chant of the Odd Number,"
from The Egyptian Book of the Dead

The problem of finding the number of numbers less than a given number is as old as mathematics, and it may be as old as mankind. The earliest known reference to it occurs on a clay tablet in the library of Hammurabi I. The text reads, "I am sixteen, going on seventeen (let it please the great god). Count the number of my years, O Shamash." Shamash replies, "I have counted thy years, O king, and they are ..." Unfortunately, at this point the text has been defaced with a hex sign. It is not believed that the Babylonians ever posed the question again.

The next to consider the problem were the Egyptians. According to Herodotus, the priests of Memphis worked on the matter for several generations. Their approach was typically Egyptian, i.e., empirical. They made a table of the numbers \( N \) and the number of numbers less than \( N \) (which we will henceforth call \( N_L \)), and tried to find a pattern. Their table went up to 152,386,917; we have reproduced a portion of it below. As you can see, there is no clear pattern:

\[
\begin{matrix}
N_L & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16
\end{matrix}
\]

Herodotus writes that the priests at last despaired of ever finding a solution, and committed suicide en masse by attempting to mate with crocodiles (see "Genetic Engineering in Ancient Egypt," Scientific American, October, 1980).

In the Sixth Century, B.C., Pythagoras conjectured that \( N_L \) would always be a small whole number. His guess has proved to be true whenever \( N \) is a small whole number. However, it receives little attention nowadays.

The first real breakthrough came in 301 B.C. In that year, a student in Plato's Academy, named Brown, devised a system for finding \( N_L \) which has worked for every \( N \) tried so far. Brown's system is very beautiful and elegant—a perfect example of the "aha!" insight so vital in creative work. We present it here.

Cut out many squares of paper or wood. Lay the squares out in \( N \) rows of \( N \) squares each. Remove the lower right-hand square. Now place each square from the last row at the end of one of the rows above it (one square per row). The new number of rows is \( N_L \) (see figure 2).

\[
\begin{array}{c}
\text{N=3} \\
\begin{array}{c}
\text{F:3} \\
2
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\text{N_L=2} \\
\begin{array}{c}
\text{F:3} \\
2
\end{array}
\end{array}
\]

No one knows why this algorithm works. The rows and columns seem to be symmetric. Why count one and not the other? A recent theory suggests that this surprising asymmetry may be due
to the curvature of space-time. If so, this would be a powerful confirmation of Einstein’s theory, as well as another indication of the deep interplay between modern physics and advanced mathematics (see “How to Bend Space-Time with a Butter-Knife,” The Amateur Scientist, Scientific American, January, 1970).

In 15 A.D., a Roman customs officer named Maximus Hubris found another algorithm. According to Pliny, Hubris’ technique involved the counting of certain stars. It is supposed to have been very accurate, but unfortunately it could not be used above 51° north latitude. For this reason the Emperor Caligula awarded Hubris with a research banishment to the Orkney Islands. His system and his fate are unknown.

Little progress was made on the problem during the Middle Ages. Still, those centuries did produce many ingenious puzzles relating to it, of which the following is an example (the solution will appear in next month’s column):

There are two logicians and two numbers. The first logician is told the sum of the numbers, the second is told their product. They have the following conversation:

“I don’t know the numbers.”
“I don’t know the numbers, either.”
“I still don’t know the numbers.”
“Neither do I.”

What are the numbers?

The next great advance came in 1963. G. P. Feldspar, then a graduate student at UCLA, was in desperate need of a thesis topic. With only months to go before graduation, he decided to count his toes. He accomplished this task with the help of special functions, which he created for the occasion, called “little piggy.” The actual construction of a little piggy is too complicated to describe here; suffice it to say that one uses a little piggy by making it go “wee wee wee all the way home,” and that this latter process entails finding $N_L$ for many $N$. Thus, Feldspar was forced to find a reliable method for computing $N_L$.

We will not attempt to give an outline of Feldspar’s algorithm (a full description can be found in his thesis, “Please Give Me a PhD”). Instead, we will illustrate its use in a delightful magic trick.

Take three apples and shuffle them. While your back is turned, have a friend take out one apple, note it, and replace it. Which apple was picked?

Set the apples in a row and mentally number them 1, 2, and 3. Tear a small hole in the skin of the second apple and fold the skin back to make a flap. With one vigorous motion, thrust a soda straw through this hole. Blow through the straw. Stop blowing when you feel out of breath. Take the other two apples and lodge them in your ears. Continue blowing. Stop when you feel out of breath again. You will have grown a third eye. Find $3_L$ (it happens to be 2). Write the answer under the skin flap (don’t show it to your friend, though). Repeat the process twice, using each apple as the second apple. Now fix all five eyes on your friend. He will tell you which apple he picked.

Feldspar’s algorithm depends on a very simple idea: $N_L < N$. In English, this says that the number of numbers less than $N$ is itself less than $N$. At first sight this statement may appear self-referential, as, e.g., in the Russell Paradox. There is, however, no paradox—though Feldspar does acknowledge a debt to the Gödel Incompleteness Theorem. Repeated applications of Feldspar’s Inequality produce $N_L$ in no more than $N^{\frac{1}{2}}$ steps.

(It should be noted that Feldspar’s work has recently stimulated research outside of pure mathematics. See “Why Toes Are Ticklish,” Scientific American, June, 1987.)

Two new developments are now causing great excitement.

Dr. Yves Steinitz, professor of the philosophy of mathematics at the Sorbonne, has created a scheme for finding $N_L$ which uses the principles of structuralist philosophy. His method is
unfortunately very subtle, and even to begin to hint at a sketch of it would take more space than this column allows. The interested reader will find it explained in Steinitz's book, _Twelve (The Architecture of Arithmetic, Vol. XII)._  

J. Landrieu Sunbelt, of the Playtex Long Lines Lab, has conjectured that $N_L = N - 1$. With the help of a Cray-2, Sunbelt has verified his conjecture for all $N$ up to $2^{97} - 1$. He is now working on the case $N = 2^{97}$.

Needless to say, much work remains to be done on this fascinating problem.

The solutions to last month's puzzles are given below:

I. All the suspects' alibis are flawless. Somebody else must have murdered him.

II. Humpty-Dumpty has become a Klein bottle. He cannot have any tarts.

III. The German smokes Kools. The Dane reads Thomas Hardy. The short penguin is the liar.

Next month's column will concern another of the adventures of Dr. Groupoid.