1. Let \( \{a_n\} \) be a sequence of positive numbers such that \( a_n \to 0 \). Can we always partition the natural numbers into infinitely many disjoint, infinite subsets \( S_j \) (\( j = 1, 2, 3, \ldots \)) such that, for each \( j \),
\[
\sum_{n : n \in S_j} a_n < \infty?
\]
Either prove that this can always be done, or exhibit a positive sequence \( a_n \to 0 \) for which it is impossible.

**Possible solution.**
The answer is YES, such a partition is always possible. Also, there’s a surprise at the end.

**Step One:** Write \( \mathbb{N} = A \cup B \), where \( A \) and \( B \) are disjoint and infinite, and \( A \) satisfies:
\[
\sum_{n \in A} a_n < \infty.
\]
We build \( A \) this way. Let \( n_1 \) be the least \( N \) such that \( a_N < 1 \). Then, having chosen \( n_1 < n_2 < n_3 < \cdots < n_k \), let \( n_{k+1} \) be the least \( N > 10 + n_k \) such that \( a_N < 1/(k+1)^2 \).

**Step Two:** Partition \( A \) into infinitely many disjoint, infinite sets \( \{ \tilde{S}_j \}_{j=1}^\infty \). Here’s one way to do it. Let \( \tilde{S}_1 \) consist of the \( n_k \)'s such that \( k \) is NOT a positive power of any prime. Let \( \tilde{S}_2 \) consist of the \( n_k \)'s such that \( k = 2^r \) for some \( r > 0 \); \( \tilde{S}_3 \) consists of the \( n_k \)'s such that \( k = 3^r \) for some \( r > 0 \); \( \tilde{S}_4 \) has the \( n_k \)'s for which \( k \) is a positive power of 5; and so on.

**Step Three:** Enumerate \( B \) in increasing order: \( B = \{b_1 < b_2 < \cdots\} \). We finish the proof by setting \( S_j = \{b_j\} \cup \tilde{S}_j \).

What’s the surprise? We didn’t need \( a_n \to 0 \), but only a subsequence of \( a_n \)'s going to 0; i.e., we only needed \( \liminf a_n = 0 \). Thus, we can construct such sets \( S_j \) even for the sequence
\[
a_n \equiv \begin{cases} (\log \log(100 + n))^{-1/100000} & \text{if } n \text{ equals a power of } 127; \\ ((n!)!)! & \text{otherwise.} \end{cases}
\]
That hardly seems fair.

2. Let \( g_n : [0, 1] \to \mathbb{R} \) be a sequence of non-negative continuous functions. Suppose that \( g_n(x) \geq g_{n+1}(x) \) for all \( n \) and for all \( x \in [0, 1] \). Suppose furthermore that \( g_n \to 0 \) pointwise. Show that the infinite series
\[
\sum_{1}^{\infty} (-1)^n g_n(x)
\]
converges uniformly on \([0, 1]\).
Possible solution.
By a previous homework (special case of Dini’s Theorem), \( g_n \to 0 \) uniformly on \([0, 1]\). But the partial sums of \( \sum (-1)^n \) are uniformly bounded. The result now follows from the Dirichlet Test for uniformly convergent series.

3. Let \( f_n : [0, 1] \mapsto \mathbb{R} \) be a sequence of continuous functions converging uniformly to some \( f : [0, 1] \mapsto \mathbb{R} \). Let \( \{x_n\} \) be a sequence of points in \([0, 1]\) such that \( x_n \to \) some \( \xi \in [0, 1] \). Prove that \( f_n(x_n) \to f(\xi) \).

Possible solution.
Let \( \epsilon > 0 \), and let \( N_1 \) be so big that \( n \geq N_1 \) implies \( |f_n(x) - f(x)| < \epsilon \) for all \( x \in [0, 1] \). Because \( f_n \to f \) uniformly, \( f \) is continuous. Let \( N_2 \) be so big that \( n \geq N_2 \) implies \( |f(x_n) - f(\xi)| < \epsilon \). Then \( n \geq \max(N_1, N_2) \) implies
\[
|f_n(x_n) - f(\xi)| \leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(\xi)| < 2\epsilon.
\]

4. Let \( f_n : [0, 1] \mapsto \mathbb{R} \) be a sequence of continuous functions converging merely pointwise to a function \( f : [0, 1] \mapsto \mathbb{R} \), and let \( \{x_n\} \) and \( \xi \) be as in problem \#3. Suppose in addition that the limit function \( f \) is continuous. Give an example to show that, even under these conditions, \( f_n(x_n) \to f(\xi) \) still might not happen.

Possible solution.
Let \( f_n(x) = n^{1/2}x(1-x^2)^n \). By the previous homework \( f_n \to 0 \) pointwise, while \( f_n(1/\sqrt{2n+1}) \to 1/\sqrt{2e} \neq 0 \).

5. Let \( \Omega \subset \mathbb{R}^d \) be open, and let \( f \) and \( g \) be differentiable functions mapping from \( \Omega \) into \( \mathbb{R} \). Define \( h(x) = f(x)g(x) \). Show that \( h \) is also differentiable and that its gradient equals \( f(x)\nabla g(x) + g(x)\nabla f(x) \).

Possible solution.
Let \( c \in \Omega \), \( v \in \mathbb{R}^d \), \( \|v\| \) small, and write:
\[
h(c+v) - h(c) = (f(c+v) - f(c))g(c+v) + f(c)(g(c+v) - g(c)). \tag{1}
\]
We need to show that this equals
\[
(f(c)\nabla g(c) + g(c)\nabla f(c)) \cdot v + o(\|v\|)
\]
as \( v \to 0 \). The second term on the right-hand side of (1) equals
\[
f(c)(\nabla g(c) \cdot v + o(\|v\|)) = f(c)\nabla g(c) \cdot v + o(\|v\|),
\]
so it’s clearly okay. The first term equals
\[
(\nabla f(c) \cdot v + o(\|v\|))g(c+v) = (\nabla f(c) \cdot v + o(\|v\|))(g(c+v) - g(c)) + (\nabla f(c) \cdot v + o(\|v\|))g(c). \tag{2}
\]
The second term on the right-hand side of (2) equals $g(c)\nabla f(c) \cdot v + o(\|v\|)$, so it’s okay. What about the first term? We’ll be done if we show that it’s $o(\|v\|)$. Recall that $g$ is \textit{continuous} at $c$, and therefore $g(c + v) - g(c) = o(1)$ as $v \to 0$. Therefore

$$o(\|v\|)(g(c + v) - g(c)) = o(\|v\|)$$

as $v \to 0$. But also, by the Cauchy-Schwarz inequality, $|\nabla f(c) \cdot v| \leq \|\nabla f(c)\|\|v\|$, so that

$$|(\nabla f(c) \cdot v)(g(c + v) - g(c))| \leq \|\nabla f(c)\|\|v\|o(1) = o(\|v\|),$$

and we’re done!