1. Rosenlicht, p. 246, # 18

2. (Volume of Revolution) Suppose that \( g : [a, b] \rightarrow \mathbb{R} \) is a positive-valued continuous function which we graph in the \( xz \)-plane by considering \( x = g(z) \). We let \( A \) be the region between the graph and the \( z \)-axis: \( A = \{(x, z) \in \mathbb{R}^2 : 0 \leq x \leq g(z), z \in [a, b]\} \). Define \( C \) to be the set obtained by revolving \( A \) around the \( z \)-axis in \( xyz \)-space: \( C = \{(x, y, z) \in \mathbb{R}^3 : (x^2 + y^2)^{1/2} \leq g(z), z \in [a, b]\} \). Show that

\[
\text{vol}(C) = \pi \int_a^b [g(z)]^2 \, dz
\]

Use the cylindrical coordinates map \( \phi(r, \theta, z) = (r \cos(\theta), r \sin(\theta), z) \) and observe that \( C \) is the image under \( \phi \) of the set \( B = \{(r, \theta, z) \in \mathbb{R}^3 : 0 \leq r \leq g(z), \theta \in [0, 2\pi], z \in [a, b]\} \). Be sure to mention how you use the change of variables theorem and Fubini's theorem.

3. Let \( A \) and \( B \) be \( n \) by \( n \) matrices with real entries. Let \( ||A|| \) denote the usual matrix norm of \( A \).
   a. Prove that \( ||AB|| \leq ||A|| \cdot ||B|| \)
   b. For any integer \( m > 0 \), show that \( ||A^m|| \leq ||A||^m \).
   c. Show that each entry of the matrix

\[
F(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k
\]

converges to a real-valued function on \( \mathbb{R} \) which is infinitely differentiable (i. e. has derivatives of all orders). You may use the facts that \( ||A + B|| \leq ||A|| + ||B|| \), \( ||cA|| = |c| \cdot ||A|| \) for any real number \( c \), and \( ||B|| \) is greater than or equal to the absolute value of any entry of \( B \).
   d. Show that if you differentiate each entry, you get \( F'(t) = AF(t) \). This is used to solve systems of differential equations.

4. Suppose that \( U \) is an open subset of \( \mathbb{R}^n \), and \( F : U \rightarrow \mathbb{R}^n \) is a function with continuous first partial derivatives on \( U \). Also suppose that the Jacobian determinant \( J_F(a) \neq 0 \) for each \( a \in U \).
   a. Show that each point \( a \in U \) is contained in an open set \( U_a \subset U \) such that \( F(U_a) \) is also an open set.
   b. Show that if \( V \) is any open set in \( U \), then \( F(V) \) is open. This is part of the open mapping theorem.