The proof is by contradiction. If $E$ is not dense in $[0, 1]$, then there is some point $a \in [0, 1]$ not in the closure of $E$, by the definition of “dense”. Then there is an open interval $J'$ around $a$ which does not contain any points of $E$, by the definition of the closure of $E$. Then $J' = J \cap [0, 1]$ is an interval containing $a$, so is nonempty and measurable, and has length $\lambda(J') = \epsilon > 0$. Now notice that $J'$ and $E$ are disjoint subsets of $[0, 1]$. So by monotonicity, $\lambda(E) \leq \lambda([0, 1] \setminus J') = \lambda([0, 1]) - \lambda(J') = 1 - \epsilon < 1$. (Note that we did not assume the $E$ is measurable, so we should be writing $\lambda^*(E)$.)

A nice way to do this problem is to use the next problem. We are assuming $X$ has finite measure, so the criterion in the next problem applies. Our hypothesis is that $f_n$ converges in measure to $f$, and that $g_n$ converges in measure to $g$. We must show that $f_n g_n$ converges in measure to $fg$. By the next problem, it suffices to show that each subsequence of $f_n g_n$ has a subsequence that converges pointwise almost everywhere. So suppose we have a subsequence $f_{k_n} g_{k_n}$. It follows from the definition that $f_{k_n}$ also converges to $f$ in measure. By the next problem, $f_{k_n}$ has a subsequence $f_{j_n}$ that converges to $f$ almost everywhere. Now consider $g_{j_n}$, which converges to $g$ in measure. By the next problem again, this has a subsequence $g_{i_n}$ which converges to $g$ almost everywhere. Now except on two sets of measure zero, we have that $f_{i_n}$ converges to $f$ and $g_{i_n}$ converges to $g$ pointwise, and we conclude that $f_{i_n} g_{i_n}$ converges to $fg$ pointwise except on the union of these two sets. Since the union of these two sets still has measure zero, we can say that $f_{i_n} g_{i_n}$ converges to $fg$ almost everywhere. We have found a subsequence of the given subsequence $f_{k_n} g_{k_n}$ that converges almost everywhere. This establishes the criterion of the next problem to allow us to conclude that $f_n g_n$ converges to $fg$ in measure.

Alternatively, given $\delta > 0$, first find an $M$ such that the measure of the set $A$ where $|f(x)| > M$ is less than $\delta/3$. This uses a previous result and the fact that $X$ has finite measure. Then, given an $\epsilon > 0$, find an $N$ such that the set $E_n$ where $|f(x) - f_n(x)| > 1$ has measure less than $\delta/3$ for all $n > N$. Hence $|f(x)| < M$ and $|f_n(x)| < M + 1$ and also $|f(x) + f_n(x)| < 2M + 1$ outside of the $B_n = A \cup E_n$ when $n > N$, and $\mu(B_n) < 2\epsilon/3$. Now we can choose $N' > N$ such that the set $C_n$ where $|f_n(x) - f(x)| > \epsilon/(2M + 1)$ has measure less than $\delta/3$ for all $n > N'$. Putting these together shows that, for each $n > N'$ the measure of $C_n \cup B_n$ is less than $\delta$ and outside of this set, we have $|f_n^2 - f^2| = |f_n - f||f_n + f| \leq \epsilon/(2M + 1))(2M + 1) = \epsilon$. This shows that $f_n^2$ converges to $f^2$ in measure, by definition. Using this, and $f_n g_n = (1/4)(f_n + g_n)^2 - (f_n - g_n)^2$ combined with previous results gives the desired conclusion.

First we assume that $X$ is a finite measure space and $f_n$ converges to $f$ in measure. If $f_{k_n}$ is any subsequence, it follow from the definitions that $f_{k_n}$ also converges to $f$ in measure. Applying Theorem 19.4 to $f_{k_n}$, we can conclude that it has a subsequence that converges to $f$ almost everywhere. This establishes one implication of the problem.

The second part of the proof is done by contradiction. So $X$ is still a finite measure space, but we assume that $f_n$ does not converge to $f$ in measure. Thus it is not true that for every $\epsilon > 0$ we have $\lim_{n \to \infty} \mu^*(\{x \in X : |f_n(x) - f(x)| \geq \epsilon\}) = 0$, by the definition of convergence in measure. This means that there exists an $\epsilon > 0$ such that $\lim_{n \to \infty} \mu^*(\{x \in X : |f_n(x) - f(x)| \geq \epsilon\}) \neq 0$. We fix this $\epsilon$. To say that the limit is not zero means that it is not true that for every $\epsilon' > 0$ there exists an $N_{\epsilon'}$ such that $\mu^*(\{x \in X : |f_n(x) - f(x)| \geq \epsilon\}) < \epsilon'$ for all $n \geq N_{\epsilon'}$, by the definition
of limit of a sequence. Thus there exists an $\epsilon'$ such that no such $N_{\epsilon'}$ exists. We fix this $\epsilon'$. Set $k_1 = 1$. Since $N_{\epsilon'} = 1$ does not satisfy the condition above, there is an integer $k_2 > 1$ such that $\mu^*(\{x \in X : |f_{k_2}(x) - f(x)| \geq \epsilon\} > \epsilon'$. Then since $N_{\epsilon'} = k_2$ also does not satisfy the condition, there is an integer $k_3 > k_2$ such that $\mu^*(\{x \in X : |f_{k_3}(x) - f(x)| \geq \epsilon\} > \epsilon'$. Proceeding in this manner, we obtain a subsequence $f_{k_n}$ such that $\mu^*(\{x \in X : |f_{k_n}(x) - f(x)| \geq \epsilon\} > \epsilon' \text{ for all } n$. We claim that this subsequence has no subsequence converging to $f$ almost everywhere, which will complete the proof. So suppose we take an arbitrary subsequence $f_{j_n}$. It still has the property that $\mu^*(\{x \in X : |f_{j_n}(x) - f(x)| \geq \epsilon\} > \epsilon' \text{ for all } n$. Thus $\lim_{n \to \infty} \mu^*(\{x \in X : |f_{j_n}(x) - f(x)| \geq \epsilon\} \neq 0$. This shows that $f_{j_n}$ does not converge in measure to $f$, by the definition of convergence in measure. By the contrapositive of Theorem 19.5, this shows that $f_{j_n}$ does not converge to $f$ almost everywhere, and completes the proof.