We proved in class as a corollary to thm 22.4, that $f$ is integrable if and only if it is measurable and $|f|$ is integrable. By definition

$$\int_E f \, d\mu = \int_X f \chi_E \, d\mu.$$ 

Since the $E_n$ are disjoint, we have $\sum_{n=1}^N \chi_{E_n} \leq \chi_E$. Thus

$$0 \leq \sum_{n=1}^N |f| \chi_{E_n} = |f| \sum_{n=1}^N \chi_{E_n} \leq |f| \chi_E \leq |f|.$$ 

First note that all of these functions are measurable since $E$ and each $E_n$ are measurable sets, and the product of measurable functions is measurable. By sandwiching (Theorem 22.6), we see that all of these functions are integrable. This implies by the corollary mentioned above, that $F_N = \sum_{n=1}^N f \chi_{E_n}$ is integrable and is dominated by the integrable function $|f|$. We can then apply the Lebesgue Dominated Convergence Theorem to get the result.

One direction follows immediately from Theorem 22.4 and Theorem 21.7. For the other direction, if $x$ is not in the exceptional set of measure zero, $f_n(x)$ gives a decreasing sequence as $n$ goes to infinity. Since it is bounded below by 0, it has a limit, which we define to be $f(x)$. For $x$ in the exceptional set, we define $f(x) = 0$. So $f(x) \geq 0$ for all $x$. Then we can apply the Lebesgue Dominated Convergence Theorem with the dominating function $f_1$ to conclude that $\int f \, d\mu = \lim \int f_n \, d\mu$, and this is 0, by assumption. Thm. 22.7 now says that $f = 0$ almost everywhere. Since $f(x)$ was defined to be the limit of $f_n(x)$ almost everywhere, we now know that $\lim_{n \to \infty} f_n(x) = 0$ almost everywhere.

p. 175, # 12

a. You have done the majority of this in p. 175, # 5, which shows $\sigma$-additivity.

b. Since part (a) shows that $\nu$ is a measure on $\Lambda$, then the sets in $\Lambda$ are $\nu$-measurable sets. This means $\Lambda \subset \Lambda_\nu$. If $f$ is zero on a $\nu$-measurable set $A$, then $\nu(A) = 0$.

For the counterexample, start with a non-$\mu$-measurable subset $B$, such as in Vitali’s example in $[0,1]$, with $\mu = \lambda$. (We cannot directly evaluate $\nu(B)$ if $B$ is not $\mu$-measurable, because we cannot integrate over a non-measurable set.) Choose a measurable set $A$ containing $B$, (e.g. $A = [0,1]$) and choose an integrable function which is zero on $A$ (e.g. $\chi_{[2,3]}$). Then the subset $B$ of $A$ has $\nu^*$-measure 0, so is $\nu$-measurable by Theorem 14.4, but it is not $\mu$-measurable.

c. We must show that $\Lambda_\nu \subset \Lambda_\mu$. Suppose that $E \in \Lambda_\nu$, so $E$ is $\nu$-measurable. By Theorem 15.11, there exists an $E'$ in $\Lambda_\mu$ containing $E$ such that $\nu(E') = \nu(E)$. We know that $E'$ is also $\nu$-measurable from part (b). So we can say that $\nu(E' - E) = \nu(E') - \nu(E) = 0$ by additivity. Let $F = E' - E$, which is $\nu$-measurable since $E'$ and $E$ are. By 15.11 again, there exists $F'$ in $\Lambda_\mu$ containing $F$ such that $\nu(F') = \nu(F) = 0$. Since $\nu(F') = 0$ and $F'$ is $\mu$-measurable, we have

$$0 = \nu(F') = \int_{F'} f \, d\mu.$$ 

Since $f \geq 0$, we conclude by Thm. 22.7 that $f = 0$ almost everywhere in $F'$. By the assumption, the set on which $f = 0$ has $\mu$-measure 0, and it contains $F'$. Thus $\mu^*(F') = 0$. Now $F \subset F'$, so $\mu^*(F) = 0$. By Theorem 14.4, $F$ is $\mu$-measurable. Finally $E = E' - F$ is $\mu$-measurable since $E'$ and $F$ are.

d. One easily checks that the equality holds for $g = \chi_A$. By linearity, it then holds for step functions. By taking limits and using the Lebesgue dominated convergence theorem, it holds for all $g$ which are $\nu$-integrable.