Math 333  Take-Home Midterm Examination
J. Sands  Due 10/30/12

This is an open-book examination. You may quote from the textbook, refer to material presented in class, and make use of homework problems you have done. However, you are instructed to adhere to a strict honor system, not consulting with anyone and not referring to any books other than our textbook, nor to any notes other than your own notes for this course. Style and clarity of exposition are important elements to be considered in your solutions.

DO ALL 3 PROBLEMS. PROBLEM 3 IS ON THE BACK OF THIS SHEET

1. (The counting measure on \( \mathbb{N} \))

Let \( X = \mathbb{N} \), let \( S \) be the collection of all finite subsets of \( \mathbb{N} \).

a) Show that \( S \) is a semiring.

b) Show that the \( \sigma \)-algebra generated by \( S \) is \( S = \mathcal{P}(\mathbb{N}) \).

Now define \( \mu : S \to \mathbb{R} \) by \( \mu(A) = |A| \), the cardinality (number of elements) of \( A \).

c) Show that \( \mu \) is a measure on the semiring \( S \).

Let \( \mu^* \) be the outer measure generated by \( \mu \).

d) Describe (with proof) the value of \( \mu^*(B) \) for each \( B \subset \mathbb{N} \).

e) Show that the collection of all \( \mu^* \)-measurable subsets of \( \mathbb{N} \) is \( \Lambda = \mathcal{P}(\mathbb{N}) \).

f) With the measure \( \mu^* \), is \( \mathbb{N} \) a finite measure space? Is it a \( \sigma \)-finite measure space? Explain.

Now consider functions on \( X = \mathbb{N} \). Notice that a function \( f : \mathbb{N} \to \mathbb{R} \) may be viewed as a sequence \( a_n = f(n) \).

g) Give an example (with proof) of a function on \( \mathbb{N} \) which is not a simple function on \( \mathbb{N} \).

h) Give an example (with proof) of a function on \( \mathbb{N} \) which is a simple function but not a step function.

i) Prove that every function on \( \mathbb{N} \) is measurable.

j) Suppose that \( f \geq 0 \). Prove that \( f \) is an upper function if and only if the series \( \sum_{n=1}^{\infty} f(n) \) converges.

k) Suppose that \( f \) is an arbitrary function on \( \mathbb{N} \). Prove that \( f \) is integrable if and only if the series \( \sum_{n=1}^{\infty} f(n) \) converges absolutely. When \( f \) is integrable, show that \( \int_{\mathbb{N}} f \, d\mu = \sum_{n=1}^{\infty} f(n) \).

2) Let \( S = \{[a, b) : a \leq b \} \) and let

\[
g(x) = \begin{cases} 
  x^3 & x \leq 0 \\
  1 & x > 0 
\end{cases}
\]

Define \( \mu : S \to \mathbb{R} \) by \( \mu([a, b)) = g(b) - g(a) \).

a) Show that \( \mu \) is a measure on \( S \). (You may use Example 13.6 from the text.)

Let \( \mu^* \) be the outer measure generated by \( \mu \).

b) Explain why the following sets are measurable. Then compute \( \mu^* \) for each. Justify your answers.

\([-3, -2), [-2, -1), [-1, 0), [0, 1), [1, 2] \]

c) Let \( f(x) = \max(x + 3, 0) \). Show that \( f \) is integrable and that \( \int_{\mathbb{R}} f \, d\mu \geq 8 \). Prove a stronger inequality if you can.
3) Do problem 7, page 118 in the text, and the following additions:
   c) Show that if $f$ is a measurable function on $X$, then the restriction $f|_E$ of $f$ to $E$ is a measurable function on $E$.
   d) If $h$ is a function on $E$, we define the extension of $h$ by zero to be
      \[
      \tilde{h}(x) = \begin{cases} 
      h(x) & x \in E \\
      0 & x \notin E 
      \end{cases}
      \]
      Show that $h$ is a measurable function on $E$ if and only if $\tilde{h}$ is a measurable function on $X$.
   e) Show that $h$ is an integrable function on $E$ if and only if $\tilde{h}$ is an integrable function on $X$.
   f) If $h$ is integrable on $E$, show that $\int_E h\,d\nu = \int_X \tilde{h}\,d\mu$. 