NAME: Solutions

Math 21 F Test III 11/12/14

1. The Mean Value Theorem.
   a. Suppose that \( f(x) \) is a function defined for all \( x \) in the interval \([3, 9]\), and \( f(3) = 8 \), \( f(9) = 2 \). If we can use the Mean Value Theorem, what does it tell us about this function?

   \[
   \frac{f(9) - f(3)}{9 - 3} = \frac{2 - 8}{6} = -\frac{6}{6} = -1.
   \]

   It tells us that there is a \( c \in (3, 9) \) with \( f'(c) = -1 \).

   b. What else do we need to know about the function \( f(x) \) before we can use the Mean Value Theorem?

      It must be differentiable on \((3, 9)\) and continuous on \([3, 9]\).

2. Indeterminate Forms. Evaluate the limits.
   a. \( \lim_{x \to 0} \frac{\sin(6x)}{\sin(2x)} \)

      \[
      8 \qquad = \lim_{x \to 0} \frac{6 \cos(6x)}{2 \cos(2x)} = \frac{6 \cos(0)}{2 \cos(0)} = \frac{6(1)}{2(1)} = 3
      \]

      So use L'Hopital.

   b. \( \lim_{x \to \infty} \frac{\ln(x^2 + 1)}{x} \)

      \[
      8 \qquad = \lim_{x \to \infty} \frac{e^x (x + 1)}{1} = \lim_{x \to \infty} \frac{e^x}{x + 1}
      \]

      Still \( \frac{\infty}{\infty} \), so use L'Hopital again.

      \[
      8 \qquad = \lim_{x \to \infty} \frac{e^x}{xe^x} = \frac{1}{1} = 1
      \]

3. Newton’s Method. Use Newton’s Method to find a better approximation to a root of \( x^3 + x + 1 = 0 \) starting with \( x_1 = -1 \).

   \[
   x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = -1 - \frac{f(-1)}{f'(-1)}
   \]

   \[
   f(-1) = (-1)^3 - 1 + 1 = -1
   \]

   \[
   f''(x) = 3x^2 + 1 \quad f''(-1) = 3(-1)^2 + 1 = 4
   \]

   So \( x_2 = -1 - \frac{-1}{4} = -1 + \frac{1}{4} = \frac{-3}{4} \).
4. **Graphing.**
   For the function \( f(x) = \frac{x^2 - 1}{x^3} \) find each of the following items:
   a. The domain of \( f \).
      \[ \forall x \neq 0 \]

   b. The \( x \)-intercepts of the graph of \( y = f(x) \).
      \[ x^2 - 1 = 0 \quad \text{so} \quad x = \pm 1 \]
      \(-1, 0) \text{ and } (1, 0)\]

   c. What kind of symmetry does the graph of \( y = f(x) \) have?
      \( f(-x) = \frac{(-x)^2 - 1}{(-x)^3} = \frac{x^2 - 1}{-x^3} = -f(x) \)
      \( f(x) \) is odd, the graph has **symmetry about the origin**

   d. The asymptotes of the graph of \( y = f(x) \).
      \( y \to \infty \) when \( x \to 0^- \) since \( \text{denom} \to 0^- \), \( \text{num} \to -1 \).
      \( x = 0 \) is vertical asymptote
      \[ \lim_{x \to \infty} \frac{x^2 - 1}{x^3} = \lim_{x \to \infty} \frac{x^2}{x^3} = \lim_{x \to \infty} \frac{1}{x} = 0 \]
      \( y = 0 \) horizontal asymptote

   e. \( f'(x) \).
      \[ f(x) = x - 1 - x^{-3} \quad \text{so} \quad f'(x) = -x^{-2} + 3x^{-4} \]
      \[ = \frac{-x^2 + 3}{x^4} \]

   f. Critical numbers for \( f(x) \).
      \( f'(x) \) undefined for \( x = 0 \)
      \[ f'(x) = 0 \quad \text{for} \quad x = \pm \sqrt{3} \]

   g. Intervals where \( f(x) \) is increasing.
      Increasing on \((-\sqrt{3}, 0) \text{ and } (0, \sqrt{3})\).

   h. Points where \( f(x) \) has a local maximum.
      \[ \text{local maximum at} \quad x = \sqrt{3} \]
      \[ y = \frac{2\sqrt{3}}{8} = \frac{\sqrt{3}}{4} \]
i. Points where $f(x)$ has a local minimum.

$$x = -\sqrt{3}, \quad y = \frac{-2\sqrt{3}}{3}$$

j. $f''(x)$.

$$f'(x) = -x^2 + 3x - 4 \quad \Rightarrow \quad f''(x) = 2x^3 - \frac{12}{x^5}$$

$$f''(x) = \begin{cases} 
\frac{2x^2 - 12}{x^5} & \text{if } x > 0 \\
\text{undefined} & \text{if } x = 0 \\
\frac{2x^2 - 12}{x^5} & \text{if } x < 0
\end{cases}$$

k. Intervals on which $f(x)$ is concave up.

$$f''(x) \text{ undefined at } x = 0$$

$$f''(x) = 0 \text{ for } 2x^2 - 12 = 0 \text{ or } x^2 = 6 \Rightarrow x = \pm \sqrt{6}$$

l. Points of inflection of $f(x)$.

$$x = -\sqrt{6}, \quad y = \frac{-5}{6\sqrt{6}} = \frac{-5\sqrt{6}}{36}$$

$$x = \sqrt{6}, \quad y = \frac{5\sqrt{6}}{36}$$

m. The graph of $y = f(x)$, showing all of the features described above.
5. Optimization.
A cardboard box with a square base and no top must be made so that the volume of the box is 32 cubic inches. Combined areas of the four sides and the bottom. What is the minimum total area of cardboard required to make the four sides and the square bottom? Be sure to show how you check that you have found a minimum and not a maximum.

1) \( y \)
2) Minimize total area \( A = 4xy + x^2 \)
3) volume is 32 \( V = x^2y \)
   \( \frac{y = \frac{32}{x^2}}{50} \)

Substituting: \( A = 4x\left(\frac{32}{x^2}\right) + x^2 \)
   \( = 128x^{-1} + x^2 \).

4) \( A' = -128x^{-2} + 2x = 0 \)
   \( 2x = 128x^2 \)
   \( 2x^3 = 128 \)
   \( x^3 = 64 \) so \( x = 4 \) is critical pt.

Check \( A'' = 256x^{-3} + 2 > 0 \) for \( x = 4 \) \( \rightarrow \) minimum.

\( \text{min area is } A(4) = \frac{128}{4} + 4^2 \)
   \( = 32 + 16 = 48 \text{ in}^2 \)