2) Let $E$ be a metric space, and suppose that $p \in E$ and $r$ is a strictly positive real number. Show that the closed ball $\overline{B}(p, r)$ satisfies the definition of a closed subset of $X$.

4) Fill in the details of the following proof leading to a (different from the text) definition and two properties of the lim sup of a bounded sequence. (We will use the traditional notation "sup" for "l.u.b.")

Suppose that $a_1, a_2, a_3, ...$ is a bounded sequence of real numbers.

a. Definition of lim sup

The bounded sequence of real numbers $a_1, a_2, a_3, ...$ has a least upper bound because...

We define $b_1 = \sup\{a_1, a_2, a_3, \ldots\}$ and $b_2 = \sup\{a_2, a_3, a_4, \ldots\}$ and in general $b_n = \sup\{a_n, a_{n+1}, a_{n+2}, \ldots\}$

Then $b_1 \geq b_2$ because ...

Similarly we have $b_n \geq b_{n+1}$ for each $n$. The sequence $b_1, b_2, b_3, ...$ is bounded because...

The sequence $b_1, b_2, b_3, ...$ converges because...

Let $b = \lim_{n \to \infty} b_n$. Notice that $b = \lim_{n \to \infty} \sup\{a_n, a_{n+1}, a_{n+2}, \ldots\}$, so $b$ is called the lim sup of the bounded sequence $a_1, a_2, a_3, ...$. This is written $b = \limsup a_n$
b. A basic property of the lim sup

We now show that for any number greater than the lim sup, there are only finitely many terms in the sequence that exceed that value.

As above, we use \( b = \limsup a_n \). Suppose that \( \epsilon > 0 \). Then there is an \( N \) such that \( b + \epsilon > b_n \) for all \( n \geq N \) because ...

Then in particular \( b + \epsilon > b_N \). Now \( b_N \geq a_n \) for all \( n \geq N \) because ...

Thus \( b + \epsilon > a_n \) for all \( n \geq N \) because ...

So there are only a finite number of \( a_n \) above or equal to \( b + \epsilon \).

c. The lim sup as the limit of a subsequence

We now show that there is a subsequence of \( a_1, a_2, a_3, \ldots \) that converges to \( b = \limsup a_n \).

First there exists an \( N_1 \) so that \( b - 1 < b_{N_1} \) because ...

Thus \( b - 1 \) is not an upper bound for \( \{ a_{N_1}, a_{N_1+1}, a_{N_1+2}, \ldots \} \) because...

Thus \( b - 1 < a_{n_1} \) for some \( n_1 \geq N_1 \). Next, there exists an \( M \) so that \( b - \frac{1}{2} < b_n \) for all \( n \geq M \) because ...

So we may choose \( n = N_2 = M + n_1 \) and get \( b - \frac{1}{2} < b_{N_2} \). Thus \( b - \frac{1}{2} \) is not an upper bound for \( \{ a_{N_2}, a_{N_2+1}, a_{N_2+2}, \ldots \} \), and \( b - \frac{1}{2} < a_{n_2} \) for some \( n_2 \geq N_2 > n_1 \). Similarly we can continue to find an increasing sequence \( n_1, n_2, n_3, \ldots \) such that \( b - \frac{1}{k} < a_{n_k} \) for each \( k \).

Now we show that the subsequence \( a_{n_1}, a_{n_2}, a_{n_3}, \ldots \) converges to \( b \), which was defined to be the lim sup of \( a_1, a_2, a_3, \ldots \). Given \( \epsilon > 0 \), we can choose an \( N \) such that \( 1/N < \epsilon \) because...

Then for all \( k > N \) we have \( a_{n_k} > b - \epsilon \), because ...

So only a finite number of terms in the subsequence are equal to or below \( b - \epsilon \). There are also only a finite number of terms in the subsequence equal to or above \( b + \epsilon \) because...

Thus the subsequence converges to \( b \) because ...