Study Guide for Final Exam

The final exam will be cumulative, but with more emphasis on the materials after Exam 3.

For materials before Exam 3, the study guides can be taken as those three exams themselves: the final exam questions on those materials will not go beyond problems on those three tests (except for very minor modifications).

For materials after Exam 3, the study guide is below.

Chapter 6

1. **Know the procedure to determine the absolute extrema of functions on a closed interval (Sec. 6.1)**

   Example: Determine the absolute maximum and minimum of the function
   \( f(x) = x^2 - 8 \ln x \) on the interval \([1, 4]\).

   Solution:

   (1) Determine critical numbers:

   \[ f'(x) = 2x - \frac{8}{x} \]

   Setting \( f'(x) = 2x - \frac{8}{x} = 0 \) gives two critical numbers \( x = \pm 2 \). But \( x = -2 \) is not in the interval \([1, 4]\) and thus is ignored.

   (2) Calculate the functional values at critical numbers and end points: \( x = 1, 2, 4 \):

   \[ f(1) = 1^2 - 8 \ln 1 = 1, \]

   \[ f(2) = 2^2 - 8 \ln 2 \approx -1.5452, \]

   \[ f(4) = 4^2 - 8 \ln 4 \approx 4.9096. \]

   Thus the absolute maximum is 4.9096, which occurs at \( x=4 \), and absolute minimum is \(-1.5452\), which occurs at \( x=2 \).

2. **Know how to solve applied problems (applications of extrema) in Sec. 6.2.**

   Example: A fence must be built to enclose a rectangular area of 200 square feet. Fencing material costs $2.00 per foot for the two sides facing north and south and $4.00 per foot for the other two sides. Find the cost of the least expensive fence.
Solution: Let $x$ be the length facing north and south, then the length of the other two sides is $200/x$. The total cost of the fence is then

$$C(x) = 2x \cdot 2.00 + 2 \cdot \frac{200}{x} \cdot 4.00 = 4x + \frac{1600}{x}$$

$$C'(x) = 4 - \frac{1600}{x^2}$$

Setting $C'(x) = 4 - \frac{1600}{x^2} = 0$, we get a single critical number $x = 20$ feet. Plugging this $x$ value into the cost function, we get the minimal cost as $160$.

3. **Know how to perform implicit differentiation (Sec. 6.4)**

Example: find $dy/dx$ for $x^2e^y + y^2 = x^3$

Solution: (below prime means derivative with respect to $x$)

$$x^2e^y + y^2 = x^3$$

$$\Rightarrow (x^2e^y)' + y'^2 = 3x^2 \Rightarrow 2xe^y + x^2e^y y' + 2yy' = 3x^2$$

$$\Rightarrow (x^2e^y + 2y)y' = 3x^2 - 2xe^y \Rightarrow y' = \frac{3x^2 - 2xe^y}{x^2e^y + 2y}$$

4. **Know how to calculate related rates (Sec. 6.5)**

Example: An ice cube that is 3 cm on each side is melting at a rate of $2 \text{ cm}^3$ per minute. How fast is the length of the side decreasing?

Solution: Let $x$ be side length of this cube and $V$ its volume, then

$$V = x^3 \Rightarrow \frac{dV}{dt} = \frac{d}{dt}(x^3) \Rightarrow \frac{dV}{dt} = 3x^2 \frac{dx}{dt}.$$  

According to the problem, when $x=3$ cm, $dV/dt= -2 \text{ cm}^3$ per min. Thus

$$-2 = 3 \cdot 3^2 \frac{dx}{dt} = 27 \frac{dx}{dt} \Rightarrow \frac{dx}{dt} = \frac{2}{27} \text{ cm / min}.$$  

Thus the length of the side is decreasing at $2/27 \text{ cm/min}$. 
Chapter 13:

Know how to differentiate trigonometric functions (sinx, cosx, etc), see Sec. 13.2.

Examples:

\[(\sin x)' = \cos x, \quad (\cos x)' = -\sin x, \quad (\tan x)' = \sec^2 x,\]

\[\sin(x^2 + 1) = \cos(x^2 + 1) \cdot (x^2 + 1)' = \cos(x^2 + 1) \cdot 2x = 2x\cos(x^2 + 1) \quad \text{(chain rule used)}\]

\[\sin^3 x = [(\sin x)^3]' = 3(\sin x)^2 (\sin x)' = 3(\sin x)^2 \cos x \quad \text{(generalized power rule used)}\]

\[(e^{\cos x})' = e^{\cos x} \cdot (\cos x)' = e^{\cos x} (-\sin x) = -\sin x \cdot e^{\cos x} \quad \text{(chain rule used)}\]

\[(\sin x \cos x)' = (\sin x)' \cos x + \sin x (\cos x)' = \cos x \cos x - \sin x \sin x = \cos^2 x - \sin^2 x \quad \text{(product rule)}\]