§3.8 Mechanical and Electrical Vibrations

Second order linear equations are important because they govern a variety of physical processes.

1. Motion of a mass on a spring

\[ F_s = -k \cdot \text{elongation} \]

**Hooke's Law:**

The spring force is proportional to its elongation,

i.e. \( F_s = -k \cdot \text{elongation} \quad k \): spring constant.

When the mass stands still, then it causes an elongation \( L \) of the spring in the downward direction.

The force balance tells us

\[ mg - kL = 0. \]

Now consider the general motion of the mass.

**Newton's Law:**

\[ m \ddot{u}(t) = \text{forces acting on it} \]

1. \( F_s = -k(u + L) \)
2. \( mg \)
3. Resistive or damping force

\[ F_d \propto \dot{u}(t) \quad \text{in general} \]

\[ F_d = -\gamma \dot{u}(t) \quad \gamma \text{ : damping constant.} \]
4. External force \( F(t) \).
\[ m \ddot{u}(t) = mg - k(u + l) - \nu u(t) + f(t) \]

\[ m \ddot{u}(t) + \nu u(t) + ku = f(t) \]

- Mass
- Damping constant
- Spring constant

This is a second order linear inhomogeneous equation

**Undamped Free Vibrations**

Suppose no external force, no damping

\[ \Rightarrow F(t) = 0, \ \nu = 0 \]

\[ \Rightarrow m \ddot{u} + ku = 0 \]

\[ \Rightarrow U'' + \frac{k}{m} U = 0 \quad \text{set} \quad \frac{k}{m} = \omega^2 \quad \text{set} \]

Its general solution is

\[ U(t) = A \cos \omega t + B \sin \omega t \]

It is more convenient to write \( U(t) \) as

\[ U(t) = R \cos(\omega t - \delta) \quad \text{where} \]

\[ R \cos \delta = A \]

\[ R \sin \delta = B \]

\[ R = \sqrt{A^2 + B^2} \]

\[ \tan \delta = B/A \]

The period of the motion is

\[ T = \frac{2\pi}{\omega_0} = 2\pi \left( \frac{m}{k} \right)^{\frac{1}{2}} \]

The frequency \( \omega_0 = \sqrt{\frac{k}{m}} \) is called the **natural frequency** of the vibration.
Damped Free Vibrations

If damping is included but no external force, then
\[ m u'' + \gamma u' + ku = 0 \]

The characteristic equation
\[ \lambda^2 + \frac{\gamma}{m} \lambda + \frac{k}{m} = 0 \]
\[ \Rightarrow \lambda_{1,2} = \frac{1}{2m} \left( -\gamma \pm \sqrt{\frac{\gamma^2}{4m^2} - \frac{4km}{m^2}} \right) \]

1) When \( \frac{\gamma^2}{4m^2} > \frac{4km}{m^2} \), \( \lambda_1 \) and \( \lambda_2 \) are real and not equal.
\[ \Rightarrow u(t) = A e^{\lambda_1 t} + B e^{\lambda_2 t} \]
Also note that \( \lambda_1 < 0, \lambda_2 < 0, \)
\[ \Rightarrow u \text{ exponentially decays with no oscillations} \]

\[ \text{overdamped motion} \]

2) When \( \frac{\gamma^2}{4m^2} = \frac{4km}{m^2} \), \( \lambda = \lambda_2 = -\frac{\gamma}{2m} \) real and repeated roots
\[ \Rightarrow u(t) = (A + Bt) e^{-\frac{\gamma}{2m} t} \]
\[ \Rightarrow u \text{ exponentially decays with no oscillations} \]

3) When \( \frac{\gamma^2}{4m^2} < \frac{4km}{m^2} \), then
\[ \lambda_{1,2} = -\frac{\gamma}{2m} \pm \frac{\sqrt{4km - \gamma^2}}{2m} i \]
\[ \text{are two complex conjugate roots} \]
Write \[ \mu = \frac{\sqrt{4km - \gamma^2}}{2m} \]
\[ \Rightarrow u(t) = e^{-\frac{\gamma}{2m} t} (A \cos \mu t + B \sin \mu t) \]
\[ = R e^{-\frac{\gamma}{2m} t} \cos (\mu t - \phi) \]
\[ \text{exponentially decays with oscillations} \]
Although this motion is not periodic, $\mu$ is still the frequency with which it oscillates back and forth. So $\mu$ is called the quasi-frequency.

$$T_d = \frac{2\pi}{\mu} : \text{ quasi-period}$$

When $\nu << 1$,

$$\mu \approx \left(1 - \frac{\nu^2}{3\kappa m}\right)\omega_0$$

So the effect of small damping is to reduce slightly the frequency of the oscillation. This makes sense.

---

**Electrical Vibrations**

![Diagram of electrical circuit with components R, C, L, and E(t)]

$Q =$ charge

$$LQ'' + RQ' + \frac{1}{C}Q = E(t)$$

The same equation as for the mechanical vibrations.
83.9 Forced Vibrations

Now consider the case where the external force is applied to a spring-mass system:

\[ m u'' + n u' + ku = F_0 \cos \omega t \]  \hfill (1)

1. First we consider the system without damping:

\[ m u'' + ku = F_0 \cos \omega t \]  \hfill (2)

The general solution of the corresponding homogeneous eqn. is:

\[ u_{ht}(t) = c_1 \cos \omega t + c_2 \sin \omega t. \quad \omega^2 = \frac{k}{m}. \]

If \( \omega = \omega_0 \), then a particular solution of (2) is:

\[ u_{pt}(t) = \frac{F_0}{m(\omega^2 - \omega_0^2)} \cos \omega t \]

\[ \Rightarrow \text{the general solution of Eq. (1) is} \]

\[ u(t) = c_1 \cos \omega t + c_2 \sin \omega t + \frac{F_0}{m(\omega^2 - \omega_0^2)} \cos \omega t \]

\[ \uparrow \text{the solution is always bounded and quasi-periodic.} \]

But what if \( \omega = \omega_0 \)?

\[ \Rightarrow u(t) = \frac{F_0}{m(\omega^2 - \omega_0^2)} \cos \omega t \cos \omega_0 t + \cos \omega_0 t \]

In this case, a particular solution of Eq. (1) is:

\[ u_{pt}(t) = \frac{F_0}{2m\omega_0} t \sin \omega_0 t \]

\[ \Rightarrow \text{the general solution of Eq. (1) is} \]

\[ u(t) = c_1 \cos \omega t + c_2 \sin \omega t + \frac{F_0}{2m\omega_0} t \sin \omega_0 t \]

\[ \uparrow \text{a linearly growing term} \]

So \( \omega_0 \to \infty \), \( u(t) \to \infty \).

It is called resonance, which always happens when the driving frequency is equal to the natural frequency.
A daily life example: a swing

Now we consider the general case with damping included:

\[ m u'' + v u' + k u = F_0 \cos \omega t \]

**Homogeneous solution:**

\[ u_h(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} \]

\[ r_1 = \frac{v}{2m} \left( -1 \pm \sqrt{\frac{v^2 - 4km}{m^2}} \right) \]

A particular solution is

\[ u_p(t) = \frac{F_0 m (\omega^2 - \omega_0^2)}{m^2 (\omega^2 - \omega_0^2) + v^2 \omega^2} \cos \omega t + \frac{F_0 v \omega}{m^2 (\omega^2 - \omega_0^2) + v^2 \omega^2} \sin \omega t \]

\[ = R \cos (\omega t - \delta) \]

where

\[ R = \frac{F_0}{\sqrt{m^2 (\omega^2 - \omega_0^2)^2 + v^2 \omega^2}} \]

\[ \tan \delta = \frac{v \omega}{m (\omega^2 - \omega_0^2)} \]

\[ \Rightarrow \text{the general solution of the equation is:} \]

\[ u(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} + R \cos (\omega t - \delta) \]

Since \( \text{Re}(r_1) < 0, \ \text{Re}(r_2) < 0, \)

\[ \omega \to \infty \quad c_1 e^{r_1 t} + c_2 e^{r_2 t} \to 0 \]

\[ \Rightarrow \quad u(t) \to R \cos (\omega t - \delta) \]

\[ \text{steady state solution or forced solution.} \]
When damping is included, we do not have resonance anymore.

At what \( \omega \) is the steady state amplitude the largest?

\[
K = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \omega^2}} = \frac{F_0}{m\sqrt{(\omega_0^2 - \frac{2k^2 - \nu^2}{2m^2})^2 + \omega^2 - (\frac{2k^2 - \nu^2}{2m^2})^2}}
\]

So when \( \omega = \omega_{\text{max}} = \frac{2k^2 - \nu^2}{2m^2} = \omega_0 - \frac{\nu^2}{2m^2} \),

the amplitude of the forced motion is the largest.

As \( \gamma \to 0 \), \( \omega_{\text{max}} \to \omega_0 \Rightarrow \text{resonance} \).

**Example 1.** Solve \[
\begin{align*}
\dddot{u} + 0.125\ddot{u} + u &= 3\cos 2t \\
\dot{u}(0) &= 2, \quad u(0) = 0
\end{align*}
\]

Homogeneous equation \( \dddot{u} + 0.125\ddot{u} + u = 0 \)

\[\lambda^3 + 0.125\lambda^2 + 1 = 0\]

\[\lambda = -0.0625 \pm 0.9980i\]

A particular solution is \( u_p(t) = 0.9965\cos(2t + 0.08314) \).

The general solution is \[
\begin{align*}
\dot{u}(t) &= e^{-0.0625t} \left( c_1 \cos 0.9980t + c_2 \sin 0.9980t \right) \\
&\quad + 0.9965 \cos(2t + 0.08314)
\end{align*}
\]

\[
\begin{align*}
\dot{u}(0) &= 2, \\
\dot{u}(0) &= 0 
\end{align*}
\Rightarrow \begin{align*}
\begin{cases}
0.125c_1 + 2.0069 = 2 \\
0.9980c_2 = 0.0625c_1 + 2.0069 \sin 0.08314
\end{cases} \Rightarrow c_1 \approx 1.0069, \quad c_2 \approx 2.2389
\end{align*}
\]

the solution is \[
\begin{align*}
u(t) &= e^{-0.0625t} \left( 1.0069 \cos 0.9980t + 0.2239 \sin 0.9980t \right) \\
&\quad + 0.9965 \cos(2t + 0.08314)
\end{align*}
\]
The solution of \( u'' + 6.125u' + u = 3 \cos t \), \( u(0) = 2 \), \( u'(0) = 0 \).