Section 4: Newton's Method and the Julia Sets

A map is an iteration scheme. One of the most important iteration schemes is
Newton's method for finding the roots of an algebraic equation. In general, Newton's
method works impressively fast (with quadratic convergence). But it requires a good
initial guess, which normally needs to be close to one of the roots. Otherwise strange
things may happen.

First, let us consider the simple example:

Suppose \( x_c \) is a critical point of \( f \) (i.e. \( f'(x_c) = 0 \)), and we apply Newton's method to the
equation \( f(x) = 0 \). If our initial guess \( x_0 \) is equal to \( x_c \) then the iteration will diverge and
will not produce any roots. If \( x_0 \) is slightly smaller or larger than \( x_c \), the outcome of the
iteration will be drastically different. This is called sensitive dependence on the initial
conditions.

Next, we discuss Newton's method applied to the simple cubic equation

\[
z^3 = 1
\]  
(4.1)

The iteration is

\[
z_{n+1} = z_n - \frac{z^3_n - 1}{3z^2_n} = \frac{2}{3}z_n + \frac{1}{3z^2_n}
\]  
(4.2)

or

\[
f(z) = \frac{2}{3}z + \frac{1}{3z^2}
\]  
(4.3)
This map has three superattracting fixed points which are the three roots of equation (4.1):
1, $e^{\frac{2\pi}{3}}$, $e^{\frac{4\pi}{3}}$. If the initial point $z_0$ of the map (4.2) is near one of these three fixed points, $f_n(z_0)$ will converge to it.

We define the attraction basins of these fixed points as

$$A_1 = \{ z : f^n(z) \to 1 \},$$

$$A_2 = \{ z : f^n(z) \to e^{\frac{2\pi}{3}} \},$$

$$A_3 = \{ z : f^n(z) \to e^{\frac{4\pi}{3}} \}.$$

Question: What do these three sets look like?

You may expect them to divide the $z$-plane and have simple clear cut boundaries between each other, just like three pieces of pie or sectors, with each sector $120^\circ$ wide. But this expectation is unfortunately too naive. The truth is that these three sets are very intricate and delicate, and are intimately interwined. The following two graphs are computer generated. They show the sets $A_1, A_2, A_3$ and their boundaries. Also refer to the graphs to be shown in class.

![Graphs showing the basins of attraction for Newton's method applied to the equation $z^4 - 1 = 0$. Left, the basin for the solution $z = 1$ is shown in black, while right only the basin boundaries are pictured. It is an example of a Julia set.](image)
Observations:

1. Symmetry. If you rotate the $z$-plane by $120^\circ$, then $A_i$ becomes $A_2, A_2$ becomes $A_3$, and $A_3$ becomes $A_1$.

2. The closure of $A_i \cup A_2 \cup A_3$ is equal to the whole $z$-plane. Thus any point $z$ is either in $A_1, A_2, A_3$ or their boundaries.

3. The boundaries of $A_1, A_2$ and $A_3$ are the same and very exotic. If we denote $\partial A = \text{boundary of the set } A$, then

$$\partial A_1 = \partial A_2 = \partial A_3.$$  

We call this common boundary the Julia set and denote it as $J$. This is in honor of Gaston Julia (1893 - 1978), a French mathematician who made remarkable contributions to maps of rational functions.

Comment: since $\partial A_1 = \partial A_2 = \partial A_3 = J$, in the neighborhood of any point in $J$, we can find points in each of $A_i$ ($i = 1, 2, 3$). This fact excludes the possibility that the sets $A_i$ ($i = 1, 2, 3$) have simple boundaries between each other, and dictates that the boundaries have to be interwined and peculiar.

4. The Julia set is approximately self-similar. If you zoom in at any point in $J$, the micro-structure you observe is roughly similar to the whole at a reduced scale (after proper rotation).

Question: What points are in the Julia set?

1. $0$ and its preimages are in $J$. Notice that $f(0) = \infty$. Hence, $0$ is the hole leading to infinity. Clearly $0 \notin \partial A_1 \cup \partial A_2 \cup \partial A_3$. Thus $0 \notin J$. We then conclude that all the preimages of $0$ are also in $J$. Note that these preimages of $0$ are countable.

2. All the periodic points of $f$ (except the three superattracting fixed points) are in $J$. Note that these periodic points clearly do not converge to any one of the three fixed points (roots), and so they are in $J$. These periodic points are unstable since some of the nearby points belong to $A_1 \cup A_2 \cup A_3$ and thus will converge to one of the fixed points. For example, solving $f^2(z) = z$ gives us the 2-periodic points as

$$z^3 = \frac{-5 \pm i \sqrt{135}}{40}.$$ They are somewhere in $J$ and are repelling.
3. Other aperiodic points which wander about in \( J \) forever must also be in \( J \). These points are uncountable and make up the majority of \( J \).

**Additional Properties of the Julia Set:**

1. \( f(J) = J = f^{-1}(J) \). That is, the images and preimages of any point in \( J \) are still in \( J \).

2. For any \( z \) in \( J \), the preimages of \( z \) are dense in \( J \).

3. \( J = \partial A_1 = \partial A_2 = \partial A_3 \).

4. There is a dense orbit in \( J \).

5. All repelling periodic points are dense in \( J \).

6. The map \( f \) is chaotic on \( J \) (by properties 4 and 5).

The proofs of these properties (except for the first one) are not trivial and will not be covered here.

To heuristically understand the above properties, we can examine Newton's method applied to the simpler equation \( z^2 = 1 \). In this case, the map is simple enough to be understood completely, and the Julia set is just the imaginary axis. Interestingly enough, this map and the Julia set also have all the properties stated above, and we can prove them all. Read the appendix of this section for details.

The next question we address concerns the geometry of the Julia set.

**Question:** How do we explain the Julia set's shape and similarity structure?

We can do it from two different aspects:

1. **Tracing the preimages of the origin.**

   First of all, \( z = 0 \) is in \( J \), since \( f(0) = \infty \).

   In the neighborhood of \( z = 0 \), if \( z = x \) is real, then \( f^n(x) \) will remain real for all \( n \). From the graph of \( y = x^3 - 1 \) and the geometric interpretation of Newton's method, we
can heuristically show that if \( x \neq 0 \) is small, then \( f'(x) \to 1 \). Thus the real axis around 
the origin (not including it) belongs to \( A_1 \). Due to the symmetry of a 120° rotation, the 
line near \( z = 0 \) with 120° inclination angle belongs to \( A_2 \) and the one with a 
\(-120°\) angle belongs to \( A_3 \). This partially explains the three petal structure of \( J \) around \( z = 0 \). Notice 
that the neighborhood of the point \( z = 0 \) contains points from all three basins of attraction 
because of the 120° rotational symmetry.

![Diagram showing the basins of attraction](image)

Next, we trace the preimages of \( z = 0 \). Solving

\[
  f(z_{-1}) = \frac{2}{3} z_{-1} + \frac{1}{3z_{-1}} = 0,
\]

we get the three preimages \( z_{-1} = f^{-1}(0) = -2 \cdot e^{\frac{2\pi i}{3}}, -2 \cdot e^{\frac{\pi i}{3}} \) and \( -2 \cdot e^{\frac{4\pi i}{3}} \). Note that the 
inverse map \( f^{-1}(z) \) has three continuous branches, each one giving a preimage of \( z \). 
Since \( z = 0 \) is surrounded by points from all three basins of attraction, the three preimages 
stated above must also be surrounded by points from all three basins of attraction. In fact, 
their infinitesimal neighborhoods must be a scaled down version of the three petals at the 
origin.

At the next round, we trace the preimages, \( z_{-2} \), of \( z_{-1} \) by solving the equation

\[
  f(z_{-2}) = z_{-1}
\]

for \( z_{-2} \). We will get nine of them. They are marked in the graph below together with the 
three \( z_{-1}' \)s. For the reasons stated above, these nine \( z_{-2}' \)s must also be surrounded by 
points from all three basins of attraction, and their infinitesimal neighborhoods are a scaled 
down version of the three petals at the origin.

Continuing this tracing process infinitely many times, we will obtain countably 
many preimages of \( z = 0 \) which are all in the Julia set. Recall that all these preimages are
also dense in $J$, and the infinitesimal neighborhood of each preimage is a scaled down version of the neighborhood of the origin. This is why we see scaled down copies of the three petals everywhere in $J$.

Another approach to studying the geometric structure of the Julia set is by constructing its support.

2. **Constructing the support of the Julia set.**

The support of a set sketches the overall geometry of the set, just as the trunk and branches of a tree are the support of a tree. It is true that this support does not include all of the tree (the leaves, for instance). But the support draws the shape of the tree well. For our Julia set, Nauenberg and Schellnhuber (1989) noted that the preimages of the negative real axis and its $\pm 120^\circ$ rotations are its support. We discuss how it works next.

We adopt the following notations:

$$M_0 = \{ x : -\infty < x \leq 0 \}$$ is the negative real axis;

$$N_0 = \{ z : z = xe^{\frac{2\pi i}{3}}, -\infty < x \leq 0 \}$$ is a 120 degree rotation of $M_0$;

$$P_0 = \{ z : z = xe^{\frac{2\pi i}{3}}, -\infty < x \leq 0 \}$$ is a -120 degree rotation of $M_0$. 
\[ M_{-1} = f^{-1}(M_0) \] is the first preimage of \( M_0 \)

\[ M_{-2} = f^{-2}(M_0) \] is the second preimage of \( M_0 \);

etc.

Other notations \( N_{-1}, N_{-2}, \ldots, P_{-1}, P_{-2}, \ldots \) are defined similarly.

First we note that

\[ M_{-1} = \{ \text{the interval } -\infty < x \leq -2^{-\frac{1}{3}} \text{ and the two curves } z = x \pm i\sqrt{x - x^2}, 0 \leq x \leq 2^{-\frac{4}{3}} \} \]

**Proof:**

Suppose \( z = x + iy \in M_{-1} \). Then \( f(z) = \frac{2}{3}x + \frac{1}{3x^2} \in M_0 \) is a negative number.

Now \( f(z) = \frac{2}{3}(x + iy) + \frac{1}{3(x + iy)^2} = \frac{1}{3} \left( 2x + \frac{x^2 + y^2}{(x^2 + y^2)^2} + i \left[ 2y - \frac{2xy}{(x^2 + y^2)^2} \right] \right) \)

In order for it to be a negative number, we require that

\[ \frac{2xy}{(x^2 + y^2)^2} = 2y, \quad \text{and} \quad 2x + \frac{x^2 - y^2}{(x^2 + y^2)^2} \leq 0. \]

The solutions of the first equation are

\[ y = 0 \quad \text{and} \quad y = \pm \sqrt{x - x^2} \quad (x \geq 0). \]

(i) When \( y = 0 \), the second equation \( \Rightarrow 2x^3 + 1 \leq 0 \Rightarrow x \leq -2^{-\frac{1}{3}}. \)

(ii) When \( y = \pm \sqrt{x - x^2} \), the second equation

\[ \Rightarrow 2x + \frac{x^2 - y^2}{x} \leq 0 \quad \Rightarrow 3x \leq \frac{\sqrt{x - x^2}}{x} \quad \Rightarrow x \leq 2^{-\frac{4}{3}} \quad \Rightarrow \quad 0 \leq x \leq 2^{-\frac{4}{3}}. \]

The proof is completed.

These three preimages of \( M_{-1} \) are plotted in Figure 2.
Because of 120° rotation symmetry, $N_{-1}$ is the 120° rotation of $M_{-1}$, and $P_{-1}$ is the $-120°$ rotation of $M_{-1}$. They are also plotted in Figure 2. Note that these first preimages form the central "flower" of the Julia set.

At the next round, $M_{-2}$ will have 9 pieces. So do $N_{-2}$ and $P_{-2}$. They are all plotted in Figure 3. We see that a finer structure has emerged. If we continue tracing the preimages of $M_0$, $N_0$ and $P_0$ an infinite number of times, we will obtain a support of the Julia set which is a skeleton of $J$.

In particular, each of the "flowers" in this support will look like

which is almost the same as the petals in the Julia set.
Comments:

1. The support we just constructed is actually equal to
   \[ \lim_{n \to \infty} (M_{-n} \cup N_{-n} \cup P_{-n}). \]
   Thus, for any \( z \) in this support, \( f^n(z) \to \) any one of those three roots \( \Rightarrow z \in J. \)
   Therefore, this support is entirely in \( J. \)

2. This support is uncountable. In fact, the total length of \( M_{-n} \cup N_{-n} \cup P_{-n} \) (not including the three straight lines extending to infinity) increases to infinity as \( n \to \infty. \)

3. All the preimages of \( z = 0 \) are in this support.
   Reason: \( 0 \in M_{-n} \) for all \( n. \) So 0 is in this support and thus the preimages of 0 are in this support.
\{ M_{-2} \cup N_{-2} \cup P_{-2} \} \text{ for } f(z) = \frac{2}{3z} + \frac{1}{(3z^2)}
Appendix: Analysis of Newton's method applied to equation $Z^2 = 1$.

The map now is: $Z_{n+1} = \frac{1}{2} (Z_n + \frac{1}{Z_n})$, \hspace{1em} (A1)

or $f(Z) = \frac{1}{2} (Z + \frac{1}{Z})$. \hspace{1em} (A2)

It has two super attracting fixed points $\pm 1$ which are the roots of $Z^2 = 1$.

We define the attraction basins of 1 and $-1$ as

\begin{align*}
A_1 &= \{ Z : f^n(Z) \to 1 \}, \\
A_2 &= \{ Z : f^n(Z) \to -1 \}.
\end{align*} \hspace{1em} (A3)

Fact:

$A_1$ is the right half of $Z$-plane $= \{ Z : \Re(Z) > 0 \}$

$A_2$ is the left half of $Z$-plane $= \{ Z : \Re(Z) < 0 \}$.

$\partial A_1 = \partial A_2$ is the imaginary axis $= \{ Z : \Re(Z) = 0 \}$.

Proof: The technique is to employ the following variable transform

\[ Z = \frac{W_{n+1}}{W_n}, \text{ or equivalently, } W = \frac{Z^n+1}{Z^n-1}. \] \hspace{1em} (A4)

When we substitute it into the map (A1), we get

\[ \frac{W_{n+1}}{W_n} = \frac{1}{2} \left( \frac{W_{n+1}}{W_n} + \frac{W_{n}}{W_{n+1}} \right) = \frac{W_{n+1}^2}{W_n^2-1} \]

\[ \Rightarrow \quad W_{n+1} = W_n^2. \] \hspace{1em} (A5)

Thus in the $W$-plane, the Newton's method is simply (A5),

or \[ F(W) = W^2, \] \hspace{1em} (A6)

which is just a squaring map.

Map (A5) is easy to analyze. One can find that

\begin{align*}
\{ W : F^n(W) \to 0 \} &= \{ W : |W| < 1 \} \text{ which is the unit disk.} \\
\{ W : F^n(W) \to \infty \} &= \{ W : |W| > 1 \} \text{ which is the outside of the unit disk.}
\end{align*}

Recall that \[ W = 0 \Leftrightarrow Z = -1 \]

\[ W = \infty \Leftrightarrow Z = 1 \]

Furthermore, the conformal mapping (A4) maps the unit disk in $W$-plane to the left half of the $Z$-plane, and the outside of the unit disk in $W$-plane to the right half of the $Z$-plane, thus

\begin{align*}
A_1 &= \{ Z : f^n(Z) \to 1 \} = \{ W : F^n(W) \to \infty \} = \{ W : |W| > 1 \} \\
&= \{ Z : \Re(Z) > 0 \}.
\end{align*}

Similarly \[ A_2 = \{ Z : \Re(Z) < 0 \}, \]

and \[ \partial A_1 = \partial A_2 = \text{the imaginary axis} = \{ Z : \Re(Z) = 0 \}. \]

The common boundary $\partial A_1 = \partial A_2$ (now the imaginary axis) is called the Julia set and denoted as $J$. 

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Next we discuss the dynamics of the map (A1) on the imaginary axis (J).
In view of (A4) and (A5), we first study the squaring map (A5) on the unit circle (denoted as S).

Dynamics of $W \rightarrow W^2$ on $S$

For $W = e^{2\pi i \theta} \in S$, the squaring map is

$e^{2\pi i \theta} \rightarrow e^{2\pi i (2\theta)}$, or $\theta \rightarrow 2\theta$ (mod 1).

If $\theta$ is expressed as a binary number, and

$\theta = 0.a_1a_2a_3 \ldots \ldots$, where $a_k (k = 1, 2 \ldots)$ are either 0 or 1,

then this map is simply a shift map, i.e.

$0.a_1a_2a_3 \ldots \ldots \rightarrow 0.a_2a_3a_4 \ldots \ldots$.

This squaring map on $S$ has all the important properties the Newton's method for $Z^3 = 1$ does. But here these properties are easy to prove.

We return to the original map (A1). Due to the transform (A4), map (A1) is equivalent to the squaring map (A5). Thus it has the following properties on the imaginary axis - its Julia set.

1. (A1) has countably infinite periodic points which are dense on the imaginary axis;
2. The preimages of any point on the imaginary axis are dense on it;
3. There is a dense orbit on the imaginary axis;
4. (A1) is chaotic on the imaginary axis.

We see that Newton's method for even the very simple equation $Z^2 = 1$ has non-trivial dynamics.