

1 Saddle-Node Bifurcation

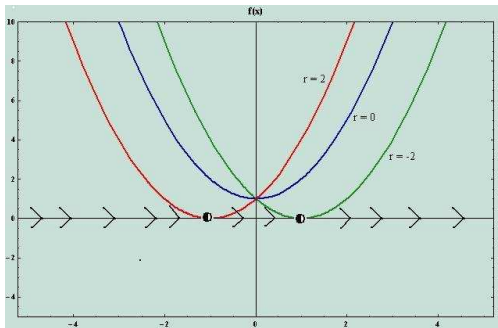
For the following differential equation, sketch all of the qualitatively different vector fields that occur as r is varied. Show that a saddle-node bifurcation occurs at the critical value $r = r_c$. Finally, sketch the bifurcation diagram of fixed points x_* versus r .

$$\dot{x} = 1 + rx + x^2$$

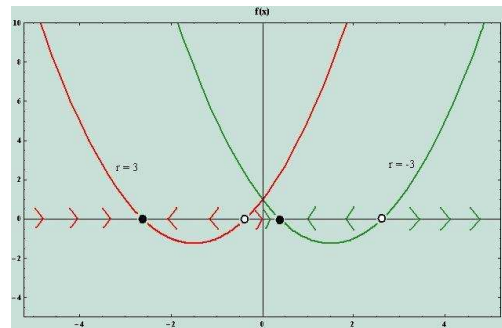
We'll start by determining the critical values r_c of r at which we have a bifurcation point. To do this we set both $f(x_*, r_c) = 1 + r_c x_* + x_*^2 = 0$ and $f_x(x_*, r_c) = r_c + 2x_* = 0$, leading to the two bifurcation points at $(x_*, r_c) = (\pm 1, \mp 2)$. From normal form analysis we can tell immediately that each of these is a saddle-node bifurcation because $\alpha = f_r(x_*, r_c) = x_* \neq 0$ for either of them.

To visualize the qualitative behavior of our dynamical system, consider the plots below. Note that the bifurcation diagram in Figure 1(c) is given by the fixed point equation:

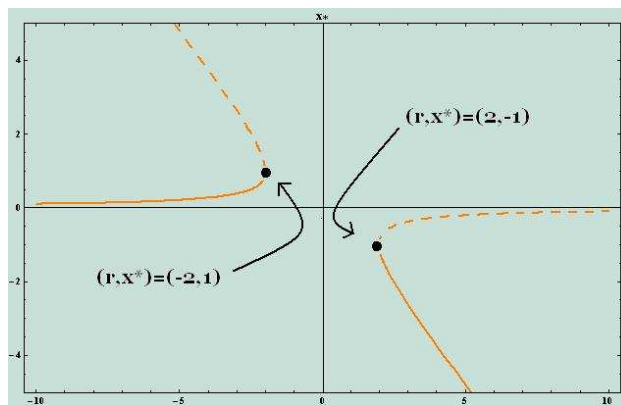
$$x_* = \frac{-r \pm \sqrt{r^2 - 4}}{2}$$



(a) Plots of $f(x)$ for $r = 2, 0, -2$, red, blue and green respectively, with associated vector field.



(b) Plots of $f(x)$ for $r = 3, -3$, red and green respectively, with associated vector fields.



(c) Bifurcation diagram x_* vs. r .

Figure 1: The vector fields fall into qualitative classes shown in plots (a) and (b): (a) for $|r| < 2$ x will go to infinity. When r reaches either bifurcation point $r_c = \pm 2$ a “semi-stable” fixed point appears such that all initial $x < x_*$ will increase but never reach x_* , while all initial $x > x_*$ will still go to infinity. (b) for $|r| > 2$ we will have two fixed points, one stable and one unstable. Whether $r < -2$ or $r > 2$ we have $x_{*stable} < x_{*unstable}$, so that the vector fields are qualitatively very similar, however we should note that in the first case both fixed points will always be positive, while in the second case they will both be negative, which can be seen clearly in the bifurcation diagram (c). Plots generated with Mathematica.

2 Pitchfork Bifurcation

For the following differential equation, sketch all of the qualitatively different vector fields that occur as r is varied. Show that a pitchfork bifurcation occurs at the critical value $r = r_c$, and classify the bifurcation as supercritical or subcritical. Finally, sketch the bifurcation diagram of fixed points x_* versus r .

$$\dot{x} = x + \frac{rx}{1+x^2}$$

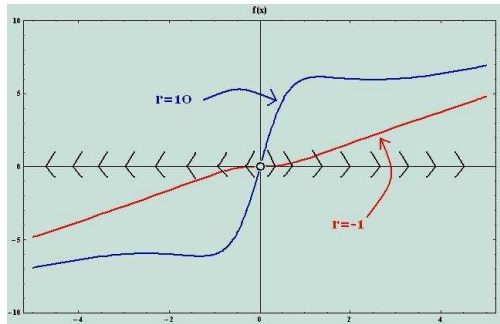
This time let's determine the fixed point equation first:

$$f(x_*) = x_* + \frac{rx_*}{1+x_*^2} = \frac{x_*(1+x_*^2) + rx_*}{1+x_*^2} = \frac{x_*[x_*^2 + (r+1)]}{1+x_*^2} = 0$$

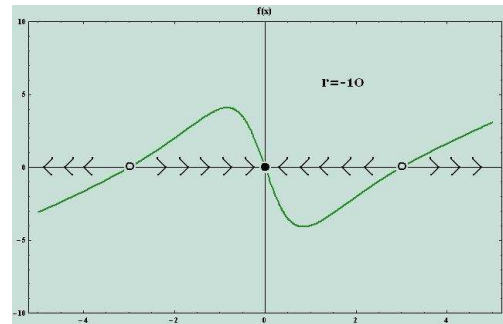
which gives

$$x_* = 0, \pm\sqrt{-(r+1)} \tag{1}$$

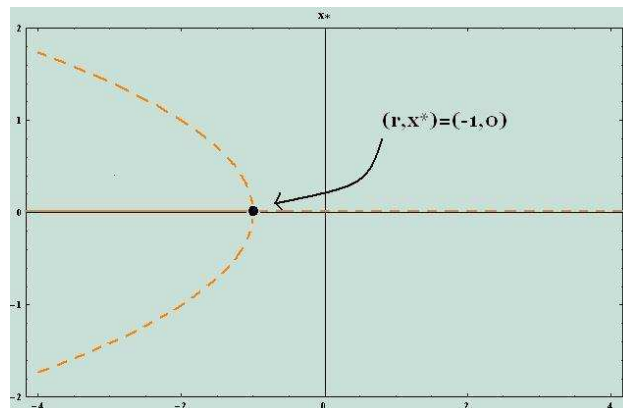
Now, it is quite clear from our fixed point equation where the bifurcation point is: $r_c = -1$. For $r \geq r_c$ we will have only 0 as a fixed point, and for $r < r_c$ we will have two fixed points in addition to $x_* = 0$ which are determined by the specific value of r . The fact that our cubic term is *destabilizing* is characteristic of a *subcritical* pitchfork bifurcation.



(a) Plots of $f(x)$ for $r = -1, 10$, red and blue respectively, with associated vector field.



(b) Plot of $f(x)$ for $r = -10$, with associated vector field.



(c) Bifurcation diagram x_* vs. r .

Figure 2: The vector fields fall into qualitative classes shown in plots (a) and (b): As r passes -1 in the negative direction, the two “bumps” on either side of the origin push through the x -axis, generating the extra two fixed points shown in (b). (c) The bifurcation diagram given by the fixed point equation (1). Plots generated with Mathematica.

3 Bifurcation Diagrams

For the vector fields below, use a computer to obtain a quantitatively accurate plot of the values of x_* vs. r , where $0 \leq r \leq 3$.

(a) $\dot{x} = r - x - e^{-x}$

At any fixed point of our system we have $f(x_*, r) = r - x_* - e^{-x_*} = 0$. An easy way to obtain the bifurcation diagram for this system (rather than solving for x_* in terms of r) is to express r as a function of x_* and simply flip the axes. Thus we have

$$r = x_* + e^{-x_*}$$

plotted with axes swapped in Figure 3. The stability of the branches is easily determined by noting that $f_x(x, r) = -1 + e^{-x} < 0$ for positive x and greater than zero for negative x . Also, by setting $f_x(x_*, r_c) = 0$ we can confirm (as the diagram shows) that our bifurcation point occurs at the fixed point $x_* = 0$ which tells us that our critical r -value $r_c = 1$.

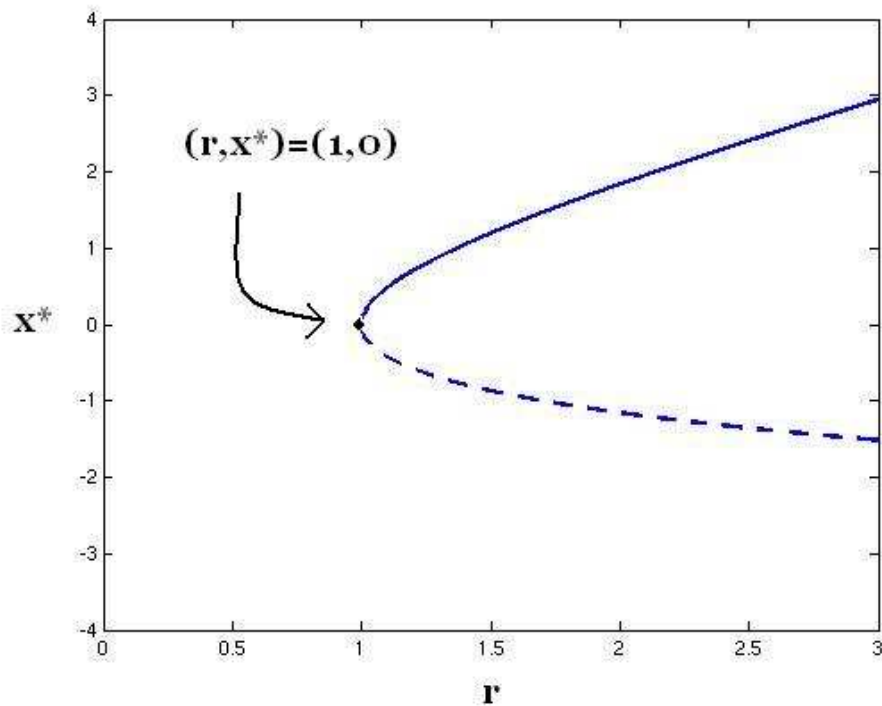


Figure 3: The bifurcation diagram shows a *saddle-node* bifurcation point at $(r, x_*) = (1, 0)$. Plot generated in MATLAB.

(b) $\dot{x} = 1 - x - e^{-rx}$

As with (a) we will solve for r in terms of x_* after setting $f(x_*, r) = 0$, and then plot with swapped axes, but there is an important difference for this system. We observe that $x_* = 0$ is *always* a fixed point, which we plot first. Then, for $x_* \neq 0$ we have

$$r = \frac{-\ln(1 - x_*)}{x_*} \tag{2}$$

We can see from this that there are no fixed points greater than or equal to 1, (which we could have guessed from the form of our differential equation as well), and also that as r goes to infinity our (second) fixed point approaches 1 asymptotically. The stability of the branches can be determined by noting that $f_x(x_*, r) = -1 + re^{-rx_*} < 0$ in two circumstances: (1) $r < 1$ and $x_* = 0$ (2) $r > 1$ and x_* is such that equation (2) is satisfied. This analysis also confirms that our critical r -value is $r_c = 1$ as shown in Figure 4.

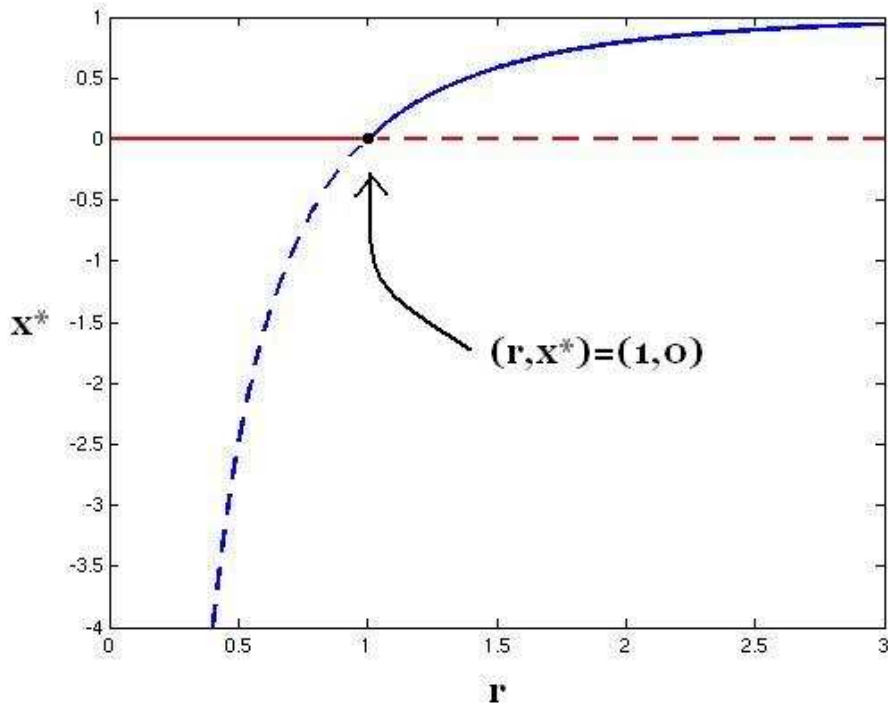


Figure 4: The bifurcation diagram shows a *transcritical* bifurcation point at $(r, x_*) = (1, 0)$. Plot generated in MATLAB.