

## Introduction

$$\text{Dynamical system} \begin{cases} \text{Discrete:} & x_{n+1} = f(x_n, t_n) & \text{maps} \\ \text{Continuous:} & \frac{dx}{dt} = f(x, t) & \text{ordinary differential equations} \end{cases}$$

Examples:

Discrete:

$$1. \quad x_{n+1} = (1 + \frac{a}{12})x_n + D \quad \begin{cases} a: \text{annual interest rate} \\ D: \text{monthly deposit} \end{cases}$$

$t$   
models your account balance at the end of each month, linear

$$2. \quad x_{n+1} = x_n^2 + C \quad \leftarrow \text{quadratic map, nonlinear}$$

If  $x_n$  is a vector variable, we will have a multi-dimensional map.

For instance,

$$3. \quad \begin{cases} x_{n+1} = y_n + 1 - ax_n^2 \\ y_{n+1} = bx_n \end{cases} \quad \begin{array}{l} \text{two-dimensional map,} \\ \text{called Henon map, also nonlinear} \end{array}$$

continuous:

$$1. \quad \frac{dx}{dt} = (t + 1)x + \sin t \quad \leftarrow \text{one-dimensional linear system}$$

If  $x$  is a vector variable, we'll have a multi-dimensional system

$$2. \quad \begin{cases} x' = y \\ y' = -\mu(x^2 - 1)y - x \end{cases} \quad \begin{array}{l} \text{two-dimensional system, nonlinear} \\ \text{note: } x' = dx/dt, \quad y' = dy/dt, \text{ etc.} \end{array}$$

$$\Leftrightarrow \quad x'' + \mu(x^2 - 1)x' + x = 0 \quad \leftarrow \text{Van Der Pol equation}$$

$$3. \quad \begin{cases} x' = \sigma(y - x) \\ y' = rx - y - xz \\ z' = xy - bz \end{cases} \quad \leftarrow \begin{array}{l} \text{three-dimensional system, nonlinear} \\ \text{Lorenz equations} \end{array}$$

The theory for linear dynamical systems is almost complete and well known.

Question: Do you know how to solve

$$\begin{cases} x_{n+1} = (1 + \frac{g}{12})x_n + D & \text{and} \\ x' = (t + 1)x + \sin t & ? \end{cases}$$

How about nonlinear dynamical systems?

Despite great advances people have made in the past thirty years, our knowledge is **still** limited.

Do you believe that even the simplest nonlinear map of all:

$$x_{n+1} = x_n^2 + C$$

is already extremely complicated? We will discuss this map in detail later in this course.

Chaos and fractals: they are the hallmarks of nonlinear dynamical systems.

They signify the complexity, unpredictability and order in the corresponding nonlinear systems.

Simply speaking, if the solutions of a nonlinear system are unpredictable, sensitive to initial conditions, we say the system is chaotic.

Fractals are beautiful and complex geometric objects which are often self-similar. Chaos usually occurs on a fractal. In some sense, fractals are the order inside chaos.

Objective of this course: is for you to learn some interesting, beautiful and useful mathematics on dynamical systems.

To many people, all mathematics is dry and uninteresting. I will show you that it is not so. Mathematics can be beautiful and artistic. The best mathematics is even philosophical.

We will also try to cover some applications of dynamical systems to science and engineering when appropriate.

Brief history of chaos and fractals

The story began three hundred year ago with Isaac Newton, when he discovered the law of gravity and invented differential equations. Equipped with calculus and laws of motion, he solved the two-body problem, i.e. the motion of the earth around the sun. Afterwards, people started to ask a natural question: how about three-body problem? for instance, the sun-earth-moon-system? It was quickly realized that the three-body problem was much harder to solve. In fact, it was impossible to obtain the solutions in explicit formulas. The situation seemed hopeless. Then two hundred years passed by until around 1900 when Henri Poincare came along. Instead of attempting to give explicit solution formulas, he

focused on the qualitative behavior of the problem. Specifically, he asked the following question: what happens to the solutions as time becomes large? He realized that the long-time solution behavior was extremely complicated. In his words,

”One will be struck by the complexity of this figure which I do not even attempt to draw. Nothing more properly gives us an idea of complication of the problem of three bodies and, in general, of all the problems in dynamics where there is no uniform integral.”

This was the first encounter with chaos.

Poincaré’s work did not receive much attention until 1960’s, when Edward Lorenz (1963) published his celebrated paper on the chaotic motion in a simplified model for weather forecast. Lorenz found in his model that small errors in the initial conditions can lead to huge differences in the later motion. This is now called sensitive dependence on initial conditions and is a trademark of chaos. This behavior practically makes long-time weather forecasting impossible. But Lorenz also showed that there was order in chaos. When he plotted the solutions in three dimensions, he obtained a beautiful butterfly pattern. He further argued that this butterfly had to have infinite layers, a structure now called a fractal. At about the same time, the fundamental work by Stephen Smale shed much light on the understanding of chaotic motion in dynamical systems.

In the 1970’s, the study of chaos flourished. Among many important contributions, those by Mitchell Feigenbaum were most influential. Feigenbaum discovered that the period-doubling bifurcation to chaos manifested in the simple logistic map was actually universal to many different dynamical systems. This was quickly verified in many experiments including fluid convection and nonlinear electronic circuits. In the 1980’s, the chaos theory was applied to practically every field of science such as physics, chemistry, biology and so on.

In a separate development, the study of fractals also flourished from the 1970’s onward. In this regard, the quadratic map played a central role. The study of this map started with Gaston Julia and Pierre Fatou around 1920. Their theoretical work revealed the tremendous complexity of this simplest map of all. In the late 1970’s, Benoit Mandelbrot discovered the celebrated Mandelbrot set and showed beautiful and intricate fractal pictures in this map. His work stunned the mathematicians and the general public alike. Mandelbrot also applied the ideas of fractals to nature such as coastal lines, and produced fascinating fractal landscapes. On the theoretical side, the important work by Dennis Sullivan, Adrien Douady and John Hubbard provided a complete understanding of the quadratic map.

Organization of the materials: see the syllabus

## Part I. Fractals and chaos in discrete dynamical systems

### § 1. Fixed points, periodic orbits, stability

Let's first consider a simple map:  $x_{n+1} = x_n^2$  where  $x$  is a complex number.

If an initial condition  $x_0$  is given, then this map will produce an infinite sequence of numbers.

$$\begin{aligned}x_1 &= x_0^2 \\x_2 &= x_1^2 = x_0^4 \\x_3 &= x_2^2 = x_0^8\end{aligned}$$

For example,

$$\begin{aligned}x_0 = 2 &\rightarrow \{2, 2^2, 2^4, 2^8, \dots\} \\x_0 = -1 &\rightarrow \{-1, 1, 1, 1, \dots\} \\x_0 = 0.5 &\rightarrow \{0.5, 0.5^2, 0.5^4, \dots\}\end{aligned}$$

The traditional way of studying a map is to try to obtain an explicit formula for  $x_n$  with  $n$  arbitrary. This is do-able for the map we are considering. Specifically

$$x_n = (x_0)^{2^n}, \quad n \geq 0$$

This approach is clean and complete. But unfortunately, it would not work for most of the maps, even very simple ones. For instance, consider a slightly different map

$$x_{n+1} = x_n^2 - 1.$$

For this map, it becomes impossible to obtain an explicit formula for the solution  $x_n$ . (If you do not believe it, I urge you to give it a try.)

All right. If explicit solution formulas are impossible to obtain, what can we do? This is exactly the dilemma Poincare faced on the three-body problem one century ago. Poincare got an idea. Instead of trying to obtain explicit solution formulas, he studied the long-time solution behavior. More specifically, for a given initial condition  $x_0$ , one studies what happens to  $x_n$  as  $n \rightarrow \infty$ . Does  $x_n$  approach a fixed value or not? etc. This new way of thinking is now called qualitative analysis as compared to the quantitative analysis of the traditional approach. In some sense, this qualitative analysis is a compromise, since it provides us less information on the solution behaviors. But the information we get is usually the most important.

Now let us reconsider the map  $x_{n+1} = x_n^2$ .

$$x_0 = 2: \{2, 2^2, 2^4, 2^8, \dots\} \Rightarrow x_n \rightarrow \infty$$

$$x_0 = 1: \{1, 1, 1, \dots\} \Rightarrow x_n \text{ fixed}$$

$$x_0 = e^{2\pi i \frac{1}{3}}: \{e^{2\pi i \frac{1}{3}}, e^{2\pi i \frac{2}{3}}, e^{2\pi i \frac{1}{3}}, e^{2\pi i \frac{2}{3}}, \dots\} \Rightarrow x_n \text{ is periodic with period 2.}$$

$$x_0 = 0.5: \{0.5, 0.5^2, 0.5^4, \dots\} \Rightarrow x_n \rightarrow 0$$

$$x_0 = 0: \{0, 0, 0, \dots\} \Rightarrow x_n = 0 \text{ fixed}$$

Notice that for this map,  $x = 0$ , and  $1$  are two special points, they are fixed points.

In general, we have the following definition.

Definition: for the map  $x_{n+1} = f(x_n)$ ,

a point  $x_*$  is called a fixed point if  $f(x_*) = x_*$ .

For instance, for the map  $x_{n+1} = x_n^2$ ,

Solve:  $x^2 = x \Rightarrow x = 0, 1$  which are fixed points for this map.

If the initial point  $x_0$  is a fixed point,  $x_n$  will stay fixed forever.

Example 2. Find the fixed points of the map  $x_{n+1} = x_n^2 - 1$ .

$$\text{Solution: } x^2 - 1 = x \Rightarrow x_* = \frac{1+\sqrt{5}}{2}$$

One important question about a fixed point  $x_*$  is its stability. i.e.

Suppose an initial point  $x_0$  is chosen to be very close to  $x_*$ , is  $x_n$  going to remain close to  $x_*$ , or go away from it?

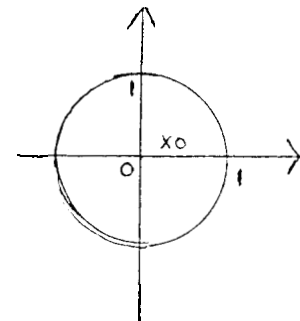
Let's consider the map  $x_{n+1} = x_n^2$  again.

It has two fixed points  $0$  and  $1$ .

First we consider  $x_* = 0$  and examine the orbit behavior if  $x_0$  is very close to  $0$ .

$$x_0 = 0.1 \Rightarrow \{0.1, 0.1^2, 0.1^4, \dots\}$$

so the orbit is attracted to  $0$ .



In fact, for  $x_0$  anywhere close to  $0$ , the orbit will approach the fixed point  $0$ .

We say  $x_* = 0$  is stable.

Next we consider  $x_* = 1$ .

Choose  $x_0 = 1.1$ ,  $\Rightarrow (1.1, 1.1^2, 1.1^4, \dots)$   
the orbit moves away from  $x_* = 1$ .

How about  $x_0 = 1.001$ ? the same.

We say  $x_* = 1$  is unstable.

Analog: pendulum. There are two stationary positions.  
Only one is stable, the other one is unstable.

Definition: Suppose  $x_*$  is a fixed point.

If for any  $x_0$  which is close to  $x_*$ ,  $x_n$  remains close to  $x_*$  for all  $n$ ,  
then we say  $x_*$  is stable.  
Otherwise, we say  $x_*$  is unstable.

Note: this is just a heuristic definition. A precise  $\epsilon - \delta$  definition can also be given, but is omitted here.

According to this definition,  $x_* = 0$  is stable and  $x_* = 1$  unstable for the map  $x_{n+1} = x_n^2$ .

For a general map  $x_{n+1} = f(x_n)$ , if  $x_*$  is a fixed point, how can we determine its stability effectively?

The answer is fairly simple.

### Linear stability analysis

Consider a general map  $x_{n+1} = f(x_n)$  with  $x_* = f(x_*)$  being a fixed point.

Suppose  $x_n$  is close to  $x_*$ , and define  $\epsilon_n \equiv x_n - x_*$ , then

$$\begin{aligned}\epsilon_{n+1} &= x_{n+1} - x_* = f(x_n) - x_* = f(x_* + \epsilon_n) - x_* \\ &\approx f(x_*) + f'(x_*)\epsilon_n - x_* \quad (\text{Taylor series expansion to order } \epsilon_n) \\ &\approx f'(x_*)\epsilon_n\end{aligned}$$

Thus,  $\epsilon_n \approx [f'(x_*)]^n \epsilon_0$

$$\Rightarrow |x_n - x_*| \approx |f'(x_*)|^n \cdot |x_0 - x_*|$$

(1) If  $|f'(x_*)| < 1$ , then  $|x_n - x_*|$  is getting smaller as  $n$  increases.

$$\Rightarrow x_n \rightarrow x_* \text{ as } n \rightarrow \infty.$$

In this case,  $x_*$  is stable, and we say  $x_*$  is attractive.

In the special case  $f'(x_*) = 0$ , we say  $x_*$  is super-attractive.

(2) If  $|f'(x_*)| > 1$ , then  $|x_n - x_*|$  is getting bigger as  $n$  increases.

$\Rightarrow x_n$  is moving away from  $x_*$ .

In this case,  $x_*$  is unstable, and we say  $x_*$  is repelling.

(3) If  $|f'(x_*)| = 1$ , we are not sure whether  $x_n$  is moving toward or away from  $x_*$ .

Thus the information is insufficient to determine  $x_*$ 's stability.

In this case, the higher order derivatives  $f''(x_*)$ , etc. are needed to determine  $x_*$ 's stability. We say  $x_*$  is indifferent in this situation.

In summary: if  $x_*$  is a fixed point of the map  $x_{n+1} = f(x_n)$ ,

$f'(x_*) = 0 \Leftrightarrow x_*$ superattractive	}	$x_*$ stable
$0 <  f'(x_*)  < 1 \Leftrightarrow x_*$ attractive		
$ f'(x_*)  = 1 \quad x_*$ indifferent		
$ f'(x_*)  > 1 \Leftrightarrow x_*$ repelling,		$x_*$ unstable

Examples:

1.  $x_{n+1} = x_n^2$ .  $f(x) = x^2$

It has two fixed points 0 and 1.

$$f'(0) = 0, \Rightarrow 0 \text{ is super attracting } \Rightarrow \text{stable}$$

$$f'(1) = 2 \Rightarrow 1 \text{ is repelling } \Rightarrow \text{unstable}$$

2.  $x_{n+1} = x_n^2 - 1$ .  $f(x) = x^2 - 1$

It has two fixed points  $x_* = \frac{1 \pm \sqrt{5}}{2}$

$$f'\left(\frac{1+\sqrt{5}}{2}\right) = 1 + \sqrt{5} \Rightarrow \frac{1+\sqrt{5}}{2} \text{ is repelling}$$

$$f'\left(\frac{1-\sqrt{5}}{2}\right) = 1 - \sqrt{5} \Rightarrow \frac{1-\sqrt{5}}{2} \text{ is also repelling}$$

Next we discuss periodic orbits and their stability.

Consider the map  $x_{n+1} = x_n^2$ ,

$$x_0 = e^{2\pi i \cdot \frac{1}{3}} \Rightarrow \{e^{2\pi i \cdot \frac{1}{3}}, e^{2\pi i \cdot \frac{2}{3}}, e^{2\pi i \cdot \frac{1}{3}}, e^{2\pi i \cdot \frac{2}{3}}, \dots\}$$

$$x_1, x_2 = x_0, x_3 = x_0 \\ \text{periodic with period 2.}$$

Definition: for a map  $x_{n+1} = f(x_n)$ ,  $x_0 \rightarrow x_1 \rightarrow x_2 \rightarrow \dots$

If  $x_p = x_0$  for some integer  $p$ ,  
then  $x_0$  is called a periodic point.  
Furthermore, if  $p$  is the smallest integer such that  $x_p = x_0$ ,  
then  $x_0$  is said to have period  $p$ .

Examples:  $x_{n+1} = x_n^2 - 1$

$$x_0 = 0 \Rightarrow \{0, -1, 0, -1, \dots\}$$

$\Rightarrow x_0 = 0$  and  $-1$  are periodic points with period 2.

How do we find the periodic points of a map in general?  
It is not hard, but often messy.

Note that if  $x_p = x_0$ , then

$$\underbrace{f \circ f \circ \dots \circ f}_{p \text{ operations}}(x_0) = x_0.$$

We define  $f^n(x) \equiv \underbrace{f \circ f \circ \dots \circ f}_n(x)$ , note:  $f^n(x) \neq [f(x)]^n$   
 $n$  compositions

then  $x_0$  is a fixed point of the map

$$y_{n+1} = f^p(y_n),$$

or it satisfies the algebraic equation

$$f^p(x_0) = x_0.$$

Solving this equation will give us the  $p$ -periodic points.

Example: Determine all the 2-periodic points of the map

$$x_{n+1} = x_n^2 - 1.$$

Solution:  $f(x) = x^2 - 1$ ,

$$f^2(x) = f(f(x)) = f(x^2 - 1) = (x^2 - 1)^2 - 1$$

We need to solve the quartic equation

$$(x^2 - 1)^2 - 1 = x$$

for 2-periodic points.

Fortunately, the algebra is not too hard in this case.

$$x^4 - 2x^2 - x = 0$$

$$x(x+1)(x^2 - x - 1) = 0$$

$$\Rightarrow x = \underbrace{0, -1}_{\text{2-periodic points}}, \underbrace{\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}}_{\text{fixed points of the map } x_{n+1} = x_n^2 - 1}.$$

fixed points of the map  $x_{n+1} = x_n^2 - 1$ .

### Stability of periodic orbits

Consider the map  $x_{n+1} = x_n^2 - 1$ ,  
it has a periodic orbit  $\{0, -1, 0, -1, \dots\}$ .  
If we start with

$$x_0 = 0.1 \Rightarrow \{0.1, -0.00, -0.0199, -0.9996, \dots, -0.0008, -1, 0, -1, \dots\}$$

accurate to four decimals.

Thus, if we start with a point close to this periodic orbit, this point will be attracted to it.  
We say this periodic orbit  $\{0, -1, 0, -1, \dots\}$  is stable and attractive.

Then we consider the map  $x_{n+1} = x_n^2$ .  
It has a 2-periodic orbit  $\{e^{2\pi i \cdot \frac{1}{3}}, e^{2\pi i \cdot \frac{2}{3}}, e^{2\pi i \cdot \frac{1}{3}}, e^{2\pi i \cdot \frac{2}{3}}, \dots\}$ .

If we start with

$$x_0 = e^{2\pi i \cdot \frac{1}{3}} + 0.1 \Rightarrow \begin{aligned} x_0 &= -0.4 + 0.8660i \\ x_1 &= -1.59 - 0.69282i \\ x_2 &= 1.0481 + 2.20322i \\ x_3 &= -4.7554 + 4.61832i \\ x_4 &= 0.2857 - 43.9239i \\ x_5 &= -1930.2 - 25.0951i \end{aligned}$$

approaches infinity

So a small initial deviation from this periodic orbit leads to large separation.

We say this periodic orbit  $\{e^{2\pi i \cdot \frac{1}{3}}, e^{2\pi i \cdot \frac{2}{3}}, e^{2\pi i \cdot \frac{1}{3}}, e^{2\pi i \cdot \frac{2}{3}}, \dots\}$  is unstable and repelling.

To study the stability of a periodic orbit in a general map

$$x_{n+1} = f(x_n),$$

notice that if  $\{x_0, x_1, x_2, \dots, x_{p-1}, x_0, x_1, \dots\}$  is a  $p$ -periodic orbit, then  $x_0$  is a fixed point of the map  $f^p(x)$ .

**Definition:** a  $p$ -periodic orbit  $\{x_0, x_1, x_2, \dots, x_{p-1}, x_0, x_1, \dots\}$  of the map  $x_{n+1} = f(x_n)$  is said to be stable if  $x_0$  is a stable fixed point of the map  $x_{n+1} = f^p(x_n)$ . Otherwise it is said to be unstable.

It is a fairly simple matter to determine the stability of a periodic orbit.

If  $x_0$  is a fixed point of the map  $x_{n+1} = f^p(x_n)$ , then

$$|(f^p)'(x_0)| < 1 \Leftrightarrow x_0 \text{ is attracting} \Leftrightarrow \text{the periodic orbit is attracting (stable)}$$

$$|(f^p)'(x_0)| > 1 \Leftrightarrow x_0 \text{ is repelling} \Leftrightarrow \text{the periodic orbit is repelling (unstable)}$$

$$|(f^p)'(x_0)| = 1 \Leftrightarrow x_0 \text{ is indifferent} \Leftrightarrow \text{the periodic orbit is indifferent}$$

$$|(f^p)'(x_0)| = 0 \Leftrightarrow x_0 \text{ is super attracting} \Leftrightarrow \text{the periodic orbit is super attracting (stable)}$$

Now the question is:  $(f^p)'(x_0) = ?$

Note that  $f^p(x) = f \circ f \circ \dots \circ f(x) = f[f[f[\dots f(x)]]]$ ,

thus  $(f^p)'(x) = f'[f^{p-1}(x)] \cdot f'[f^{p-2}(x)] \dots f'[f(x)] \cdot f'(x)$  according to the chain rule

$$\Rightarrow \underline{(f^p)'(x_0) = f'(x_{p-1}) \cdot f'(x_{p-2}) \dots f'(x_1) \cdot f'(x_0)}$$

**Example ■** Determine the stability of the periodic orbit  $\{0, -1, 0, -1, \dots\}$  in the map

$$x_{n+1} = x_n^2 - 1$$

**Solution:**  $f(x) = x^2 - 1$ ,  $f'(x) = 2x$ . Here  $p = 2$ ,  $x_0 = 0$ ,  $x_1 = -1$

$$(f^2)'(x_0) = f'(x_1) \cdot f'(x_0) = 2x_1 \cdot 2x_0 = 0$$

$\Rightarrow$  this periodic orbit is super attracting (of course stable).

Example 2. Determine the stability of the periodic orbit  $\{e^{2\pi i \cdot \frac{1}{3}}, e^{2\pi i \cdot \frac{2}{3}}, e^{2\pi i \cdot \frac{1}{3}}, e^{2\pi i \cdot \frac{2}{3}}, \dots\}$  in the map  $x_{n+1} = x_n^2$ .

Solution:  $f(x) = x^2$ ,  $f'(x) = 2x$ ,  $p = 2$ ,  $x_0 = e^{2\pi i \cdot \frac{1}{3}}$ ,  $x_1 = e^{2\pi i \cdot \frac{2}{3}}$ ,

$$(f^2)'(x_0) = f'(x_1) \cdot f'(x_0) = 2x_1 \cdot 2x_0 = 4e^{2\pi i \cdot \frac{1}{3}} \cdot e^{2\pi i \cdot \frac{2}{3}} = 4$$

$\Rightarrow$  this periodic orbit is repelling.

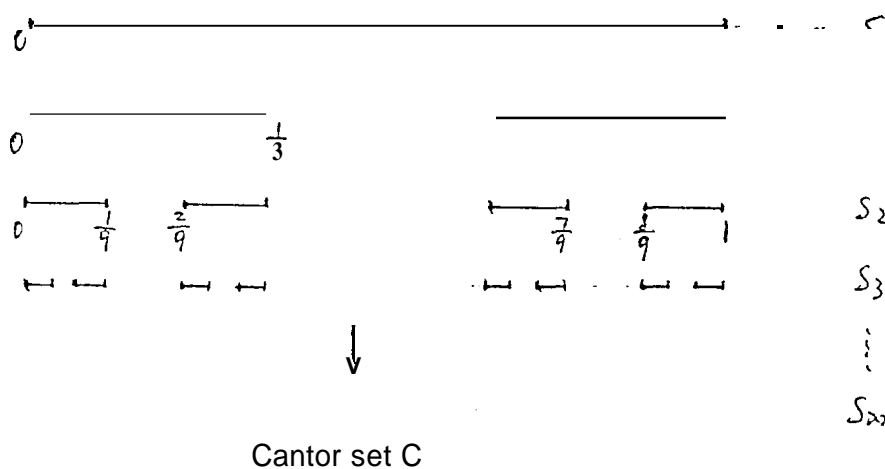
## §2. The Cantor set and the tent map

Reference: textbook Sec. 11.2.

The Cantor set is the simplest incarnation of fractals and chaos. In this section, we discuss this set and the dynamics associated with it.

### 1. The Cantor set.

Cantor set was invented in 1883 by the German mathematician Georg Cantor. It is constructed as follows:



We start with the closed interval  $S_0 = [0, 1]$ , and remove its open middle third  $(\frac{1}{3}, \frac{2}{3})$ . This produces the pair of closed intervals shown as  $S_1$ . Then we remove the open middle thirds of those two intervals to produce  $S_2$ , and so on. The limiting set  $S_\infty$  or the set of points left over, is called the Cantor set and denoted as  $C$ .

Cantor set is a strange animal. It has the following properties:

1.  $C$  is totally disconnected. It does not contain any interval.
2.  $C$  is self-similar. It contains small copies of itself at all scales. If we enlarge a small portion of it, what we see is still  $C$  except scaled down by a certain factor.
3.  $C$  is uncountable.  $C$  clearly contains all the end points of the removed open middle-third intervals, which are countable. But it contains much much more points than that. We'll prove its uncountability later in this section.

Exactly what points are in  $C$ ? To answer this questions, we introduce the ternary representatiton of a number. If a number  $x$  is in  $[0, 1]$  and has an expansion in power of  $\frac{1}{3}$ :

$$x = \frac{a_1}{3} + \frac{a_2}{3^2} + \frac{a_3}{3^3} + \dots$$

where  $a_n$  is 0, 1, or 2 for all  $n \geq 1$ , then the ternary representation of  $x$  is

$$x = 0.a_1a_2a_3\dots$$

Example 1. If the ternary representation of  $x$  is 0.1202, then

$$\text{Solution: } x_{base10} = \frac{1}{3} + \frac{2}{3^2} + \frac{0}{3^3} + \frac{1}{3^4} = \frac{46}{81}$$

Example 2. If  $x_{base3} = 0.121212\dots = 0.\bar{12}$ ,  $x_{base10} = ?$

$$\begin{aligned} \text{Solution: } x_{base10} &= \frac{1}{3} + \frac{2}{3^2} + \frac{1}{3^3} + \frac{2}{3^4} + \dots = \left(\frac{1}{3} + \frac{2}{3^2}\right) + \frac{1}{3^3} \left(\frac{1}{3} + \frac{2}{3^2}\right) + \frac{1}{3^4} \left(\frac{1}{3} + \frac{2}{3^2}\right) + \dots \\ &= \frac{\frac{1}{3} + \frac{2}{3^2}}{1 - \frac{1}{3^2}} = \frac{5/9}{8/9} = \frac{5}{8} \end{aligned}$$

Example 3. If  $x_{base3} = 0.022222\dots = 0.0\bar{2}$ ,  $x_{base10} = ?$

$$\text{Solution: } x_{base10} = \frac{0}{3} + \frac{2}{3^2} + \frac{2}{3^3} + \dots = \frac{2/3^2}{1 - 1/3} = \frac{1}{3}$$

Note that in *base 3*,  $0.0\bar{2} = 0.1$ , (both equal to  $\frac{1}{3}$  in decimal representation. Thus the ternary representation of a number is not unique. (This is true to any base representation including decimal.)

In the previous three examples, we showed how to reduce a ternary number to decimal. The reverse question is: how do we find the ternary representation of a decimal number. It is also simple: multiply the number by 3; in the product, the digit in front of the decimal point is recorded as  $a_1$ ; then drop this digit. Repeat this process in the subsequent repetitions. When this procedure is finished, the ternary representation of the number will be  $0.a_1a_2a_3\dots$ .

Example 4. Find th ternary representation for  $\frac{46}{81}$ .

$$\begin{aligned} \text{Solution: } \frac{46}{81} \times 3 &= \frac{138}{81} = 1 \frac{57}{81} \\ &\quad \uparrow \\ &\quad a_1 \\ \frac{57}{81} \times 3 &= \frac{171}{81} = 2 \frac{9}{81} \\ &\quad \uparrow \\ &\quad a_2 \end{aligned}$$

$$\frac{9}{81} \times 3 = \frac{27}{81} = 0 \frac{27}{81}$$

↑  
 $a_3$

$$\frac{27}{81} \times 3 = \frac{81}{81} = 1$$

↑  
 $a_4$  done

$$\Rightarrow \frac{46}{81} = 0.1201;$$

Example 2. Find the ternary representation for  $\frac{5}{8}$ .

Solution:  $\frac{5}{8} \times 3 = \frac{15}{8} = 1 \frac{7}{8}$

↑  
 $a_1$

$$\frac{7}{8} \times 3 = \frac{21}{8} = 2 \frac{5}{8}$$

↑  
 $a_2$

start repeating  $\frac{5}{8} \times 3 = \dots\dots$

$$\Rightarrow \frac{5}{8} = 0.121212 \dots\dots = 0.\overline{12}$$

Now, we are ready to describe what numbers are in  $C$ .

**Fact** In ternary notation, if we forbid a 1 followed by all 0s or all 2s (to make the representation unique), then the Cantor set consists of ternary numbers that have no 1s in any place.

For instance,  $0.2$ ,  $0.2022$ ,  $0.0\overline{2}$  etc. are in  $C$ .  
 $0.1202$ ,  $0.\overline{12}$ ,  $0.20122$ , etc. are not in  $C$ .

How about  $0.12222 \dots\dots = 0.1\overline{2}$  ( $= \frac{2}{3}$  in base10)?  
Remember this representation has a 1 followed by all 2s, thus is outlawed.  
The legitimate ternary representation of  $\frac{2}{3}$  is  $0.2$ , thus is in  $C$ .

The above fact is very easy to explain by recalling how the Cantor set is constructed. In the first phase when the open middle third  $(\frac{1}{3}, \frac{2}{3})$  is removed from  $[0, 1]$ , all ternary numbers whose first digit is 1 are taken out. In the second phase, all ternary numbers whose second digit is 1 are taken out. Repeating this process infinite times, what is left in the Cantor set are ternary numbers that have no 1s in any digit place.

Note: the end points of those removed open intervals are in the Cantor set  $C$  whose ternary representations are tailed by all 0s or all 2s. They make a small fraction of points in  $C$ .

Now we can prove the uncountability of  $C$ . The proof is by contradiction. If  $C$  were

countable, we could list them as follows (in ternary numbers)

$$\begin{aligned} x_1 &= 0.x_{11}x_{12}x_{13}x_{14} \dots \\ x_2 &= 0.x_{21}x_{22}x_{23}x_{24} \dots \\ x_3 &= 0.x_{31}x_{32}x_{33}x_{34} \dots \\ x_4 &= 0.x_{41}x_{42}x_{43}x_{44} \dots \end{aligned} \quad \text{where } x_{ij} \text{ s are either 0 or 2.}$$

Then we construct a new number  $r$  as

$$r = 0.x_{11}^*x_{22}^*x_{33}^*x_{44}^* \dots$$

$$\text{where } x^* = \begin{cases} 0, & \text{if } x = 2, \\ 2, & \text{if } x = 0. \end{cases}$$

Then  $r$  is different from every  $x_i$  in  $C$ .  
 But  $r$  has no 1s in any digit places  $\Rightarrow r \in C$ . } contradiction.

Thus  $C$  is uncountable.

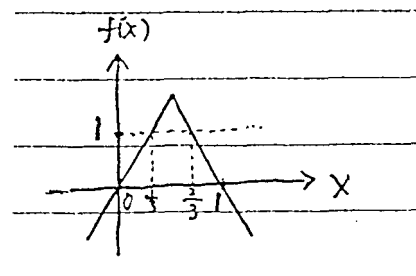
At above, we discussed the geometric and algebraic structures of the Cantor set. Now we introduce dynamics to this set.

## 2. The tent map.

Consider the map (tent map)

$$x_{n+1} = f(x_n) = \begin{cases} 2x_n, & x_n \leq \frac{1}{2}. \\ 3(1-x_n), & x_n \geq \frac{1}{2}, \end{cases}$$

where  $x_n$  is real.



For instance,

$$\begin{aligned} \text{if } x_0 = \frac{1}{3} &\Rightarrow \left\{ \frac{1}{3}, 1, 0, 0, \dots \right\}, \\ \text{if } x_0 = \frac{1}{2} &\Rightarrow \left\{ \frac{1}{2}, \frac{3}{2}, -\frac{3}{2}, -\frac{9}{2}, -\frac{27}{2}, \dots \right\}. \end{aligned}$$

Given this map, how do we study it?

Recall that we are now mostly concerned with the long-time behavior of the solution

For  $x_0 = \frac{1}{3}$ ,  $x_n \rightarrow \pm \infty$  as  $n \rightarrow \infty \Rightarrow \{x_n\}$  is bounded,  
 For  $x_0 = \frac{1}{2}$ ,  $x_n \rightarrow -\infty$ , as  $n \rightarrow \infty \Rightarrow \{x_n\}$  is unbounded, or  
 $x_n$  escapes to infinity.

In general, we want to know,

for what values of  $x_0$  does  $\{x_n\}$  remain bounded for all  $n$  ?  
 or what is the set

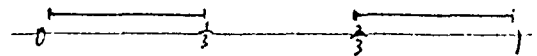
$$S = \{x_0 \mid x_n \rightarrow \pm \infty \text{ as } n \rightarrow \infty\} ?$$

To answer this question, we first examine what values of  $x_0$  make  $x_n$  escape to infinity

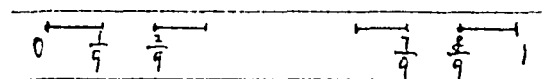
Obviously, if  $x_0 < 0 \Rightarrow x_n \rightarrow -\infty$ ;  
 if  $x_0 > 1 \Rightarrow x_1 < 0 \Rightarrow x_n \rightarrow -\infty$ .

Next we examine the unit interval  $[0, 1]$ .

(1) if  $x_0 \in (\frac{1}{3}, \frac{2}{3}) \Rightarrow x_1 > 1 \Rightarrow x_n \rightarrow -\infty$ .



(2) if  $x_0 \in (\frac{1}{9}, \frac{2}{9}) \Rightarrow x_1 \in (\frac{1}{3}, \frac{2}{3}) \Rightarrow x_n \rightarrow -\infty$ ;



if  $x_0 \in (\frac{7}{9}, \frac{8}{9}) \Rightarrow x_1 \in (\frac{1}{3}, \frac{2}{3}) \Rightarrow x_n \rightarrow -\infty$ ;  
 and so on.  $\vdots$

Now if we take away all those  $x_0$  intervals that make  $x_n$  escape to infinity, what is left is the set  $S = C$ , the Cantor set.

Thus  $C = \{x_0 \mid x_n \rightarrow \pm \infty \text{ as } n \rightarrow \infty\}$ .

The next important question is:

if  $x_0 \in C$ , will  $\{x_n\}$  be periodic or otherwise?

To answer this question, it is convenient to adopt the ternary number system, which we will do in the rest of this section.

In the ternary system, if  $x \in C$  and  $x = 0.a_1a_2a_3a_4 \dots$ ,

$$\text{then } f(x) = \begin{cases} 0.a_2a_3a_4 \dots, & \text{if } a_1 = 0, \\ 0.a_2^*a_3^*a_4^* \dots, & \text{if } a_1 = 2, \end{cases}$$

where  $a_n^* = 2 - a_n$ .

For instance,

$$f(0.0202) = 0.202, \\ f(0.2202) = 3(1 - 0.2202) = 3 \times 0.00202 = 0.0202.$$

Now we are ready to determine the periodic orbits of this map.

Example 1. Show that  $x = 0.\overline{20}$  is a fixed point of this map.

Solution:  $f(0.\overline{20}) = f(0.202020\dots) = 0.202020\dots = 0.\overline{20}$ .

Example 2. Show that  $\{0.\overline{2200}, 0.\overline{0220}, 0.0220\dots\}$  is a periodic orbit with period 2.

Solution:  $f(0.\overline{2200}) = f(0.22002200\dots) = 0.02200220\dots = 0.\overline{0220}$

$f(0.\overline{0220}) = f(0.02200220\dots) = 0.22002200\dots = 0.\overline{2200}$ .

Example 3. Show that  $\{0.\overline{220}, 0.\overline{020}, 0.\overline{200}, 0.\overline{220}, 0.\overline{020}, 0.\overline{200}, \dots\}$

and  $\{0.\overline{222000}, 0.\overline{002220}, 0.\overline{022200}, 0.\overline{222000}, 0.\overline{002220}, 0.\overline{022200}\}$

are two 3-periodic orbits of the tent map.

In general, we have the following result

Theorem 1.

*Proof:* For  $x \in C = 0.a_1a_2\dots a_k$ , define its index  $I(x) = \begin{cases} 1 & \text{if } k \text{ is odd} \\ 0 & \text{if } k \text{ is even} \end{cases}$ .  
 $I(x)$  is even  $\Leftrightarrow f^k(x)$  involves even number of flips.  
 Thus  $0.a_1a_2\dots a_{n+1}$  is periodic  $\Leftrightarrow a_{n+1} = 0$ .

- (1)  ~~$x \in C$  is a  $(n+1)$ -periodic point of the tent map if and only if  $x$  is in the form of  $0.\overline{a_1a_2\dots a_n0}$  (where the number of 2s in  $a_k$  is even) or  $0.\overline{a_1a_2\dots a_n2a_1a_2\dots a_n}$  (where the number of 2s in  $a_k$  is odd).~~  
 ~~$x \in C$  is a  $(n+1)$ -periodic point  $\Leftrightarrow x = 0.a_1a_2\dots a_n0$ .~~
- (2) The Cantor set  $C$  contains a countable infinity of Periodic orbits of the tent map, including orbits of arbitrary period.
- (3) The periodic points of the tent map are dense in  $C$ .

The proof of this theorem is easy and left as an exercise.

How about the stability of these infinitely many periodic orbits? Obviously unstable, because in the neighborhood of each periodic point in  $C$ , there are points  $x \notin C$  whose orbits escape to infinity. Actually, we can further show that all these periodic orbits are repelling since

$$|(f^p)'(x)| = |f'(x_{p-1})| \cdot |f'(x_{p-2})| \cdot \dots \cdot |f'(x_1)| \cdot |f'(x_0)| = 3^p > 1$$

By now, we have completely understood the periodic orbits in the tent map.

If  $x_0 \in C$  and is not a periodic point, what happens to its orbit?

(I) Preperiodic points.

If  $x_0 = 0.0\overline{2200} \Rightarrow (0.0\overline{2200}, 0.\overline{0220}, 0.\overline{2200}, 0.\overline{2200}, 0.\overline{0220}, \dots)$

we say  $0.0\overline{2200}$  is a preperiodic point.

Other examples of preperiodic points are:

$$x_0 = 0.00\overline{20} \Rightarrow (0.00\overline{20}, 0.0\overline{20}, 0.\overline{20}, 0.20, \dots)$$

$$x_0 = 0.\overline{220} \Rightarrow (0.2\overline{20}, 0.\overline{02}, 0.\overline{20}, 0.20, \dots)$$

$$x_0 = 0.2\overline{2200} \Rightarrow (0.22200, 0.\overline{0022}, 0.2200, 0.0220, 0.\overline{0220}, 0.\overline{2200}, \dots)$$

etc.

(2) Aperiodic points.

If  $x_0 = 0.220222022220\dots$ , then the orbit is neither periodic nor preperiodic. It is called an aperiodic point. In general, if the digits of a number are not repeating, then it is an aperiodic point of the tent map. Most of the points in  $C$  are aperiodic.

Next we discuss some other important properties of the tent map.

1. Existence of a dense orbit in  $C$ .

In other words, there is an orbit  $\{x_0, f(x_0), f^2(x_0), \dots, f^n(x_0), \dots\}$  in  $C$  which can be arbitrarily close to any point in  $C$ . This means that the orbit of  $x_0$  will visit every corner of the set  $C$ . It also means that, for any open set  $U$  in  $C$ , the images of  $U$  is dense in  $C$ .

Proof. Just take  $x_0 = 0. \underbrace{02}_{\text{all 1 blocks}} \underbrace{000202022}_{\text{all 2 blocks}} \underbrace{000003,020022,200202,220222}_{\text{all 3 blocks}} \dots$

*This number needs to be modified. If a block ends with 0 do nothing. If it ends with 2, add a 0 after it. This modification guarantees that when the operation reaches a block, the digits in this block are the way they originally are (not flipped).*

Recall that the tent map is similar to an operation which "forgets" the first digit of a number, it is easy to see that the orbit of  $x$  will be arbitrarily close to any point in  $C$ .

2. Sensitive dependence on initial conditions.

Consider two nearby initial points  $x_0^{(1)}$  and  $x_0^{(2)}$ , one in  $C$  and the other one outside of  $C$ . Then as the iteration proceeds, one orbit remains in  $C$  forever, while the other one escapes to infinity. In other words, small differences in the initial conditions will lead to vastly different orbits later on. This is the so-called sensitive dependence on initial conditions. What about two nearby initial points both in  $C$ ?

Example 1. Take

$$x_0^{(1)} = 0.00002, x_0^{(2)} = 0.000002, |x_0^{(2)} - x_0^{(1)}| = 0.00022 \text{ very small}$$

$$x_1^{(1)} = 0.0002, x_1^{(2)} = 0.00002,$$

.....

$$x_4^{(1)} = 0.2, \quad x_4^{(2)} = 0.02, \quad |x_4^{(2)} - x_4^{(1)}| = 0.22, \text{ quite large}$$

So in this example, small initial distance between the two orbits is amplified rapidly by the iteration. Again sensitive dependence on i.c.

Example 2. Take

$$x_0^{(1)} = 0.\overline{22002200}, x_0^{(2)} = 0.\overline{22002202},$$

you can show that these *two* nearby points will lead to two very different periodic orbits.

The above examples indicate that the tent map depends sensitively on initial conditions at every point in  $C$ .

## Chaos

We call the complex, sensitive and somewhat unpredictable behavior of the tent map on the Cantor set  $C$  chaotic. Chaos is a much-abused name in science. There is no uniformly accepted definition for it. Chaos in different fields of science, even just in the field of dynamical systems can mean different things for different people. Here we adopt the definition Robert Devaney proposed.

Definition:

Let  $F: M \rightarrow M$  be a map, where  $M$  is a metric space.

The map  $F$  is said to be chaotic if

1.  $F$  has sensitive dependence on initial conditions everywhere in  $M$ ;
2. There is a dense orbit in  $M$ ;
3. Periodic points of  $F$  are dense in  $M$ .

The tent map on  $C$  satisfies all the three conditions. Thus it is chaotic.

Comment: these three conditions for chaos are not independent. Actually, Banks, et. al. (1992) proved that, if the second and third conditions are satisfied, then the first one is automatically satisfied. But to emphasize the sensitive dependence of a chaotic map, we still put it in as a condition.

## Summary of this section:

The tent map is one of the only few maps whose behaviors can be understood completely and explicitly. Despite its simple, innocent looks, it is chaotic. It depends sensitively on initial conditions.

The set  $\{x_0 \mid f^n(x_0) \rightarrow \infty\}$

is the Cantor set, which is the simplest fractal.

### Additional Notes on The Cantor Set and the Tent Map

1. The preperiodic points of the tent map are of the form

$$s = 0.b_1b_2 \dots \cdot bm \overline{a_1a_2 \dots a_n0},$$

where the number of 2s in  $b_k$  is even, and the number of 2s in  $a_k$  is even,

$$\text{or, } x = 0.b_1b_2 \dots \cdot bm \overline{a_1^*a_2^* \dots a_n^*2},$$

where the number of 2s in  $b_k$  is odd and the number of 2s in  $a_k$  is even,

$$\text{or, } x = 0.b_1b_2 \dots \cdot bm \cdot a_1a_2 \dots a_n \overline{2a_1^*a_2^* \dots a_n^*0},$$

where the number of 2s in  $b_k$  is even, and the number of 2s in  $a_k$  is odd,

$$\text{or, } x = 0.b_1b_2 \dots \cdot bm \cdot \overline{a_1^*a_2^* \dots a_n^*0} a_1a_2 \dots a_n 2,$$

where the number of 2s in  $b_k$  is odd, and the number of 2s in  $a_k$  is odd.

These points are countable, because the periodic points are countable, and the preimages of a periodic point are also countable.

2. Why does the Cantor set contain points which are not the end points of the removed intervals?

Because the end points are terminating, such as 0.1, 0.2, 0.21, etc.

(The convention for unique representation is relaxed.)

But there are much more points in  $C$  which are not terminating.

3. The Cantor set and the unit interval  $[0, 1]$  have the same cardinal indices.

$$\text{Proof: } x_{(3)} = 0.a_1a_2a_3 \dots \in C \quad \begin{matrix} 2 \rightarrow 1 \\ \leftarrow 1 \end{matrix}, \quad x_{(2)} = 0.b_1b_2b_3 \dots \in [0, 1] \\ (0x = 0, 2) \quad (b_k = 0, 1)$$

This correspondence is one to one.

So in some sense, the Cantor set is just as large as the unit interval  $[0, 1]$ , even though the length of  $C$  is zero.