

Area-preserving maps (Hamiltonian Chaos)

Area-contracting maps (such as Henon map) may have strange attractors and chaos. Interestingly, area-preserving maps may also have chaos, even though no strange attractors.

The continuous counterpart of area-preserving maps is hamiltonian systems. Chaos in hamiltonian systems was first pointed out in the nineteenth century by Poincaré. But his work was not appreciated until the 1960's, when numerical computations revealed fascinating chaotic behaviors in many simple dynamical systems.

A beautiful area-preserving map was constructed and studied in 1969, again by Henon. The map is

$$\begin{cases} x_{n+1} = x_n \cos \alpha - (y_n - x_n^2) \sin \alpha \\ y_{n+1} = x_n \sin \alpha + (y_n - x_n^2) \cos \alpha \end{cases} \quad \alpha \text{ is an arbitrary parameter}$$

$$\text{or } P_{n+1} = T(P_n)$$

Properties:

1. It is invertible.

$$\begin{cases} x_n = x_{n+1} \cos \alpha + y_{n+1} \sin \alpha \\ y_n = -x_{n+1} \sin \alpha + y_{n+1} \cos \alpha + (x_{n+1} \cos \alpha + y_{n+1} \sin \alpha)^2 \end{cases}$$

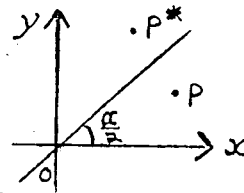
2. Symmetry:

Denote by P^* the symmetrical of a point P with respect to the straight line

$$y = \tan\left(\frac{\alpha}{2}\right)x,$$

then

$$T^{-1}(P^*) = [T(P)]^*.$$



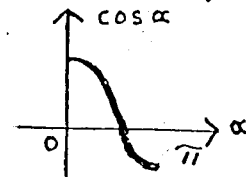
Thus if we consider a set of points obtained by repeated applications of the mapping:

$$P_0 \quad P_1 = T(P_0), \quad P_2 = T(P_1), \dots, P_n = T(P_{n-1}), \dots,$$

then, there exists a symmetrical set, obtained by starting from P_n^* and applying n times the mapping T . The reason is that

$$P_{n-1}^* = [T^{-1}(P_n)]^* = T(P_n^*).$$

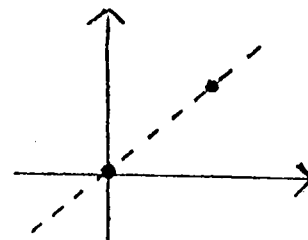
3. Some points escape to infinity. For instance, when x becomes large, $T^n(x,y) \rightarrow$ infinity.
4. If α is replaced by $2\pi - \alpha$, T will be replaced by its symmetrical with respect to the y -axis. Thus it is sufficient to consider the range of values $0 \leq \alpha \leq \pi$.



Fixed points of T^n :

- (1) $n = 1$: two fixed points

$$\begin{array}{ccc} (0,0), & \text{and} & (2 \tan \frac{\alpha}{2}, 2 \tan^2(\frac{\alpha}{2})) \\ I_{11} & & I_{12} \\ \text{always stable} & & \text{always unstable} \end{array}$$



- (2) $n = 2$: four fixed points

I_{11} , I_{12} and two complex points:

$$I_{21}: \quad (-\cot(\frac{\alpha}{2}) + i(1 + \sin^{-2}(\frac{\alpha}{2}))^{\frac{1}{2}}, \quad -1 - i \cot(\frac{\alpha}{2})(1 - \sin^{-2}(\frac{\alpha}{2}))^{\frac{1}{2}})$$

$$I_{22}: \quad (-\cot(\frac{\alpha}{2}) - i(1 + \sin^{-2}(\frac{\alpha}{2}))^{\frac{1}{2}}, \quad -1 + i \cot(\frac{\alpha}{2})(1 - \sin^{-2}(\frac{\alpha}{2}))^{\frac{1}{2}})$$

which we do not care since we only consider real values of x and y .

- (3) $n = 3$: eight fixed points

I_{11} , I_{12} , and

$$\left(\frac{-(1+\cos\alpha)+\theta}{\sin\alpha}, \quad \frac{\cos^2\alpha-(1+\cos\alpha)\theta}{\sin^2\alpha} \right)$$

$$\left(\frac{-\cos\alpha-\theta}{\sin\alpha}, \quad \frac{\cos^2\alpha-\cos\alpha-1+(1+\cos\alpha)\theta}{\sin^2\alpha} \right)$$

$$\left(\frac{-\cos\alpha-\theta}{\sin\alpha}, \quad \frac{\cos^2\alpha-\cos\alpha+(\cos\alpha-1)\theta}{\sin^2\alpha} \right)$$

with $\theta = \pm \sqrt{\cos^2\alpha - 2\cos\alpha - 1}$. We call them I_{31} , I_{32} and I_{33} for θ_+

and I_{34} , I_{35} and I_{36} for θ_-

These six fixed points exist only if

$$\cos\alpha \leq 1 - \sqrt{2}.$$

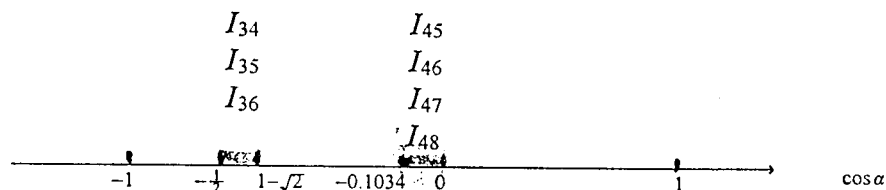
Stability: I_{31} , I_{32} and I_{33} are always unstable.

$$I_{34}, I_{35} \text{ and } I_{36} \begin{cases} \text{stable for } -\frac{1}{2} < \cos \alpha < 1 - \sqrt{2} \\ \text{unstable for } -1 < \cos \alpha \leq -\frac{1}{2} \end{cases}$$

(4) $n = 4$: 16 fixed points

$$I_{11} \quad I_{12}: \quad I_{21} \quad I_{22}$$

$$\begin{array}{ccc} \underbrace{I_{41}, I_{42}, I_{43}, I_{44}}_{\text{exist for } \cos \alpha \leq 0} & \underbrace{I_{45}, I_{46}, I_{47}, I_{48}}_{\text{exist for } \cos \alpha \leq 0} & \underbrace{I_{49}, I_{4,10}, I_{4,11}, I_{4,12}}_{\text{never real, can be neglected}} \\ \text{always unstable} & \begin{cases} \text{stable for } -0.1034 < \cos \alpha \leq 0 \\ \text{unstable for } -1 < \cos \alpha \leq -0.1034 \end{cases} & \end{array}$$



Numerical results:

Question: given an initial point P_0 , what happens to its successive iterations

$$P_0, \quad P_1 = T(P_0), \quad P_2 = T(P_1), \dots$$

$$\text{or } \{P_0, TP_0, T^2P_0, \dots, T^n P_0, \dots\} \quad ?$$

An example: $\cos \alpha = 0.24$.

$$\left\{ \begin{array}{l} \text{invariant curves} \\ \text{islands} \\ \text{scattered set:} \end{array} \right. \begin{array}{l} \text{fills some region in the plane} \\ \text{almost always escape to infinity in the end} \end{array}$$

Important fact: Every island has essentially the same structure as the whole region, on a reduced scale.

1. Sensitive dependence on i.c
2. Fractal structure
3. No attractors

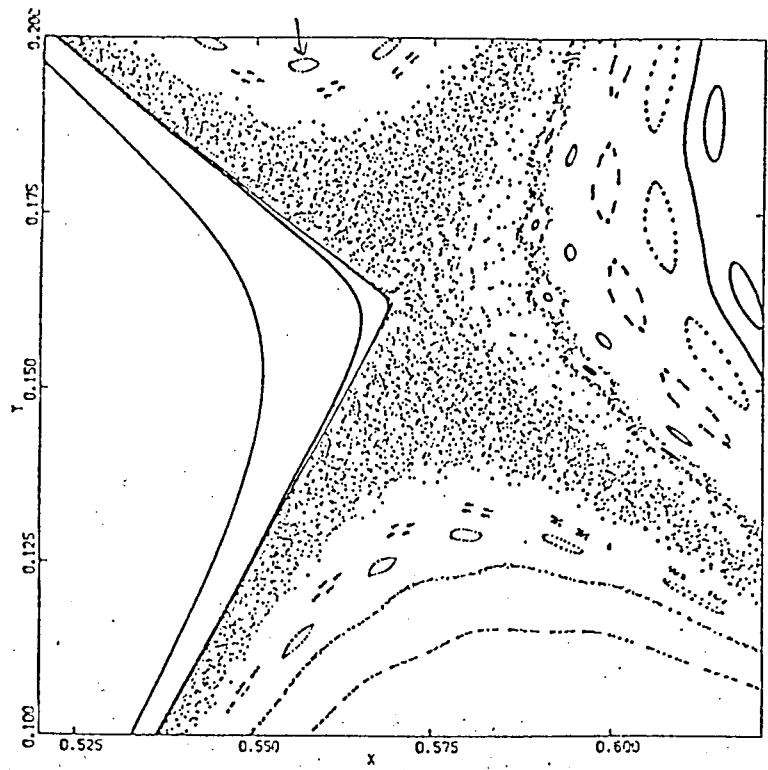


FIG. 14. $\cos \alpha = 0.24$: enlargement of the vicinity of an unstable invariant point of T^4 .

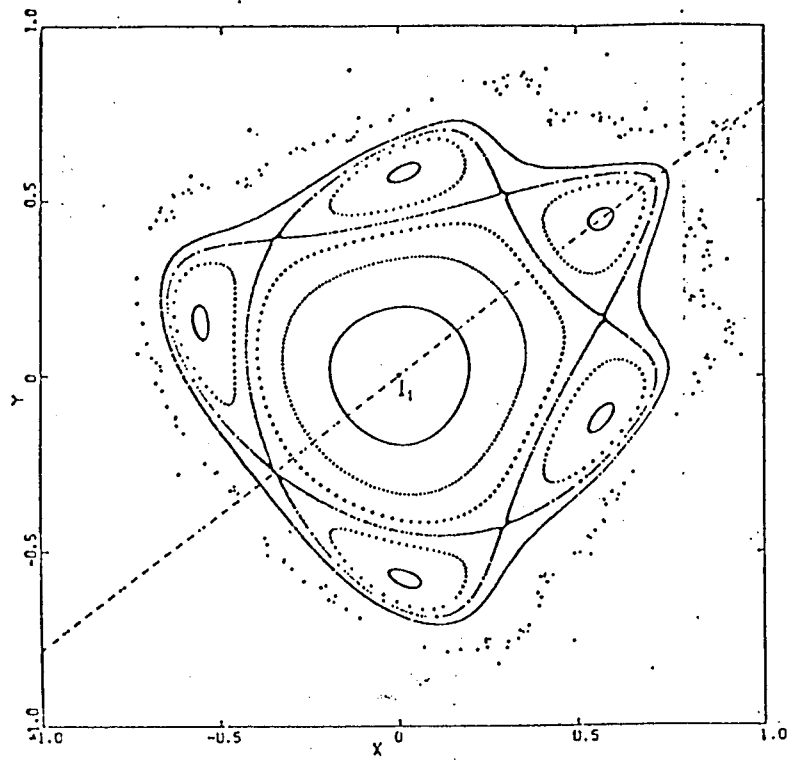


FIG. 5. $\cos \alpha = 0.24$.

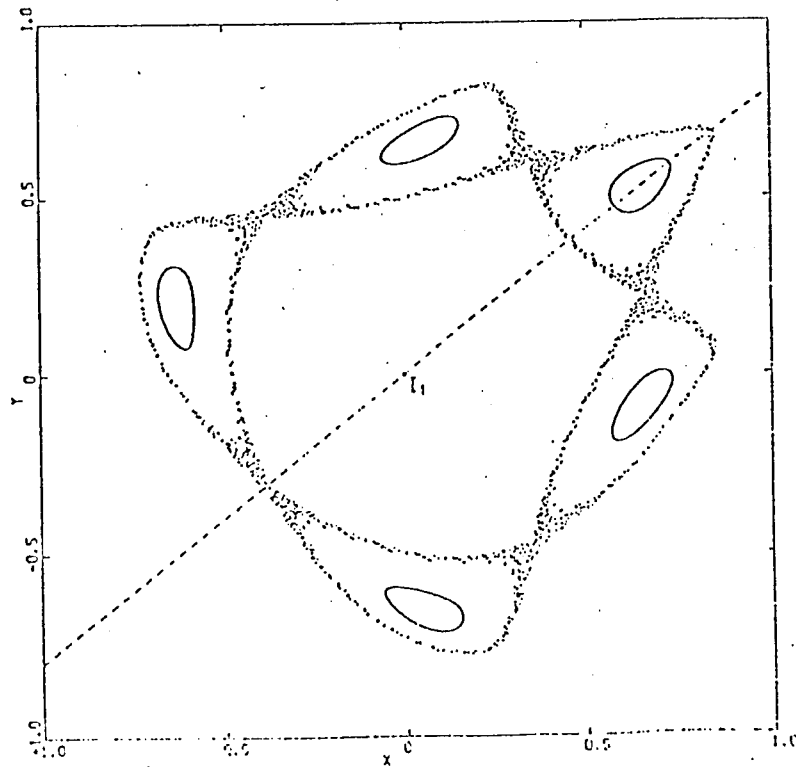


FIG. 13. $\cos \alpha = 0.22$: a scattered set surrounding five islands.

