Lorenz Equations with Other Parameter Values

So far we have concentrated on the particular parameter values

$\sigma = 10, \quad b = \frac{8}{3}$ and $r = 28$, as in Lorenz (1963).

What happens if these parameters are changed?

It turns out that the dependence of the solution structure on these parameters is very complicated. Depending on the choices of these parameters, you can find limit cycles, intermittent chaos, noisy periodicity, as well as strange attractors.

Note 1. Noisy periodicity (reference Lorenz (1980a)

Suppose that for some suitable return plane we can locate n non-overlapping connected regions $E_1, E_2, \bullet \bullet E_n$ on the return plane such that all trajectories eventually pass through these regions in cyclic order, then we say the system is "semi-periodic" with period n. Sometimes, semi-periodicity is also called noisy periodicity.

Such motion arises just below the limit points of the infinite sequences of doubling bifurcations in the Lorenz system.

For example when $197.6 < r < 214.364$ and

r just below 145 $(\sigma = 10, \quad b = \frac{8}{3})$

2. Intermittent Chaos (reference Manneville & Pomeau 1979, 1980)

The trajectories at times are almost periodic, then they wander off and behave chaotically for a while, and then return to the almost periodic state again. This process repeats infinitely.

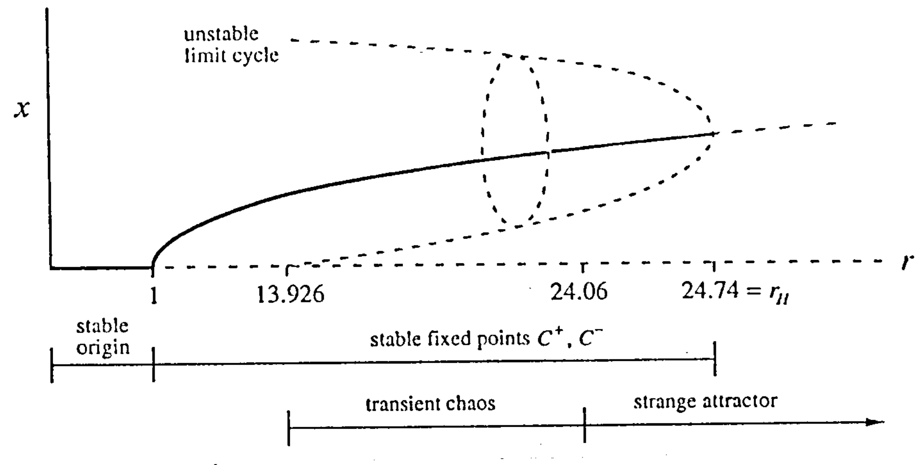
Intermittent chaos arises at the top end of a period-doubling window in the Lorenz system.

For example, when r is just above 166.07 and 100.795

$(\sigma = 10, \quad b = \frac{8}{3})$.

The Lorenz equations have three parameters σ, b and r. In this vast 3-D parameter space, much remains to be discovered. To simplify matters, most researchers have kept $\sigma = 10$ and $b = \frac{8}{3}$ while varying r.

When r is small, the solution behavior is summerized in the following figure.



Much of this picture is familiar.

Two new results:

1. At $r = 13.926$, a pair of unstable limit cycles are created. the bifurcation occured here is called homoclinic bifurcation. At this point, a "strange invariant set" is born. This set consists of a countable infinity of periodic orbits, an uncountable infinity of aperiodic orbits, and an uncountable infinity of trajectories which terminate in the origin.

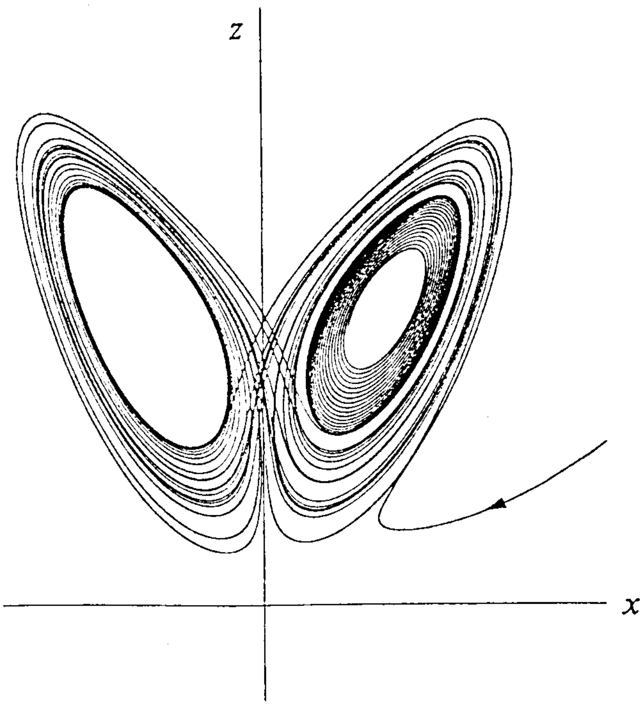
(Note: the two unstable limit cycles involved in the subcritical Hopf bifurcation at $r = 24.74$ are in this set.)

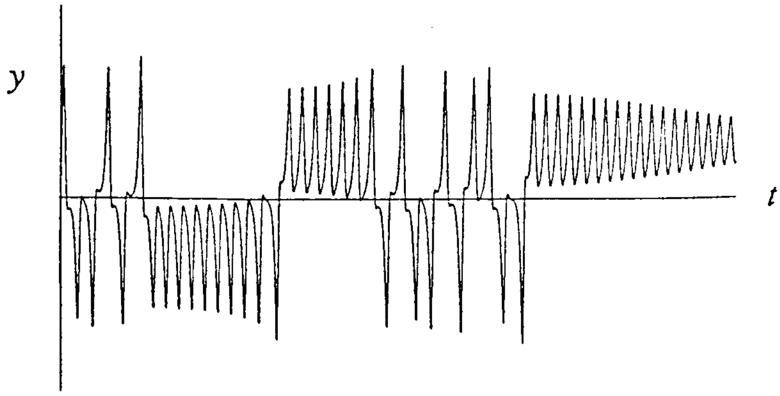
All the orbits and trajectories in this strange invariant set remain forever within a small region specified in Sparrow (1982, P21), and they are all individually unstable. This strange invariant set is not an attractor and is not numerically observable, but it generates sensitive dependence on initial conditions, i.e. two nearby trajectories within the strange invariant set eventually move far apart. Trajectories can wander on this set for a while, but eventually escape and settle down to C^+ or C^-. The time spent wandering on this set gets longer and longer as r increases. Finally, at $r = 24.06$, the time spent wandering becomes infinite and the set becomes a strange attractor (Yorke and Yorke 1979).

Demonstration: transient chaos

$$r = 21, \quad (\sigma = 10, \quad b = \tfrac{8}{3})$$

After a few tries, it is easy to find solutions like that shown below.





2. When $24.06 < r < 24.74$, stable fixed points (stable) and a strange attractor coexist. This means that the solution can be chaotic and non-chaotic, depending on the initial conditions. It also means that a large enough perturbation can knock a stable fixed point into chaos.

Solution Behaviors for Large r

When r is large, introduce $\epsilon = r^{-\frac{1}{2}}$

and the scalings:
$$\begin{cases} x = r^{\frac{1}{2}} u_1 \\ y = \frac{r}{\sigma} u_2 \\ z = r(\frac{u_3}{\sigma} + 1) \\ t = \frac{T}{r^{\frac{1}{2}}} \end{cases}$$

then the Lorenze equations become

$$\begin{cases} u_1' = u_2 - \sigma\epsilon u_1 \\ u_2' = -u_1 u_3 - \epsilon u_2 \\ u_3' = u_1 u_2 - b\epsilon(u_3 + \sigma) \end{cases}$$

At the lending order, set $\epsilon = 0_u$

$$\Rightarrow \quad \begin{cases} u_1' = u_2 \\ u_2' = -u_1 u_3 \\ u_3' = u_1 u_2 \end{cases}$$

This set of equations has two integrals

$$\begin{cases} u_1^2 - 2u_3 = 2A, \\ u_2^2 + u_3^2 = B^2. \end{cases}$$

Using these integrals, the equation $u_3' = u_1 u_2$ becomes

$$u_3' = \sqrt{2(B - u_3)(u_3 + B)(u_3 + A)}$$

The solution to this equation can be expressed as an elliptic integral and is always periodic. (Sparrow 1982).

So when r is very large, the solutions become periodic and are simple again. Numerical simulations indicate that the system has a globally attracting symmetric limit cycle for all $r > 313$. This limit cycle with $r = 350$ is plotted on the next page.

What happens for $24.74 < r < 313$?

In this wide range, the solution structure is not quite simple. For most values of r one finds chaos, but there are also iwndows of periodic behavior interspersed. The three largest windows are $99.524 < r < 100.795$; $145 < r < 166$; and $r > 214.4$. $126.4 < r < 126.55$ is another smaller one. We will describe the largest window $r > 214.4$ in some detail, and make some general comments for the other windows.

The Window $r > 214.4$

$r > 313$: a globally attracting symmetric limit cycle exists.

$r = 313$: the symmetric limit cycle loses its stability and two asymmetric stable limit cycles appear. The bifurcation is "pitchfork" like, and sometimes called "symmetry breaking".

$230 < r < 313$: two asymmetric stable limit cycles exist and they are globally attracting.

$r = 230$: each of the two asymmetric limit cycles loses its stability and a stable asymmetric limit cycle with twice of its period appears. This is a period-doubling bifurcation.

$218 < r < 230$: two stable asymmetric limit cycles exist. Their periods are roughly twice of those for $230 < r < 313$.

$r = 218$: the two asymmetric limit cycles lose their stability and two new stable asymmetric limit cycles with twice of their previous periods appear. This is another period-doubling bifurcation.

\- -

$214 < r < 218$: in this interval, a cascade of period-doubling bifurcations occurs. Each time such a bifurcation happens, the period of a limit cycle is doubled.

$r \approx 214$: the period of a limit cycle becomes infinity and the period-doubling bifurcation is terminated.

This bifurcation sequence is illustrated on a separate page, and typical limit cycles in various r intervals are also plotted.

Note: this period-doubling cascade resembles that in the logistic map. Actually, the period-doubling bifurcations occur in many dynamical and physical systems, and they are remarkably similar. This similarity has been mathematically explained.

Other periodic windows:
$$\begin{cases} 99.524 < r < 100.795 \\ 126.4 < r < 126.55 \\ 145 < r < 166 \\ etc \end{cases}$$

In these windows, the bifurcations and the solution behaviors are similar to those in the window $r > 214.4$. (See the separate page for the bifurcation diagrams.) They all feature the period-doubling bifurcation.

There are some minor differences among each other, which we will not elaborate here. (Read Sparrow (1982) for details.)

What happens outside those periodic windows?

The solutions mostly are chaotic.

Just ahead or below a periodic window, the solutions strictly speaking are chaotic, but their irregularity is not so bad and they still bear some resemblance to periodic solutions.

A bifurcation diagram over the entire r values is plotted on a separate page.

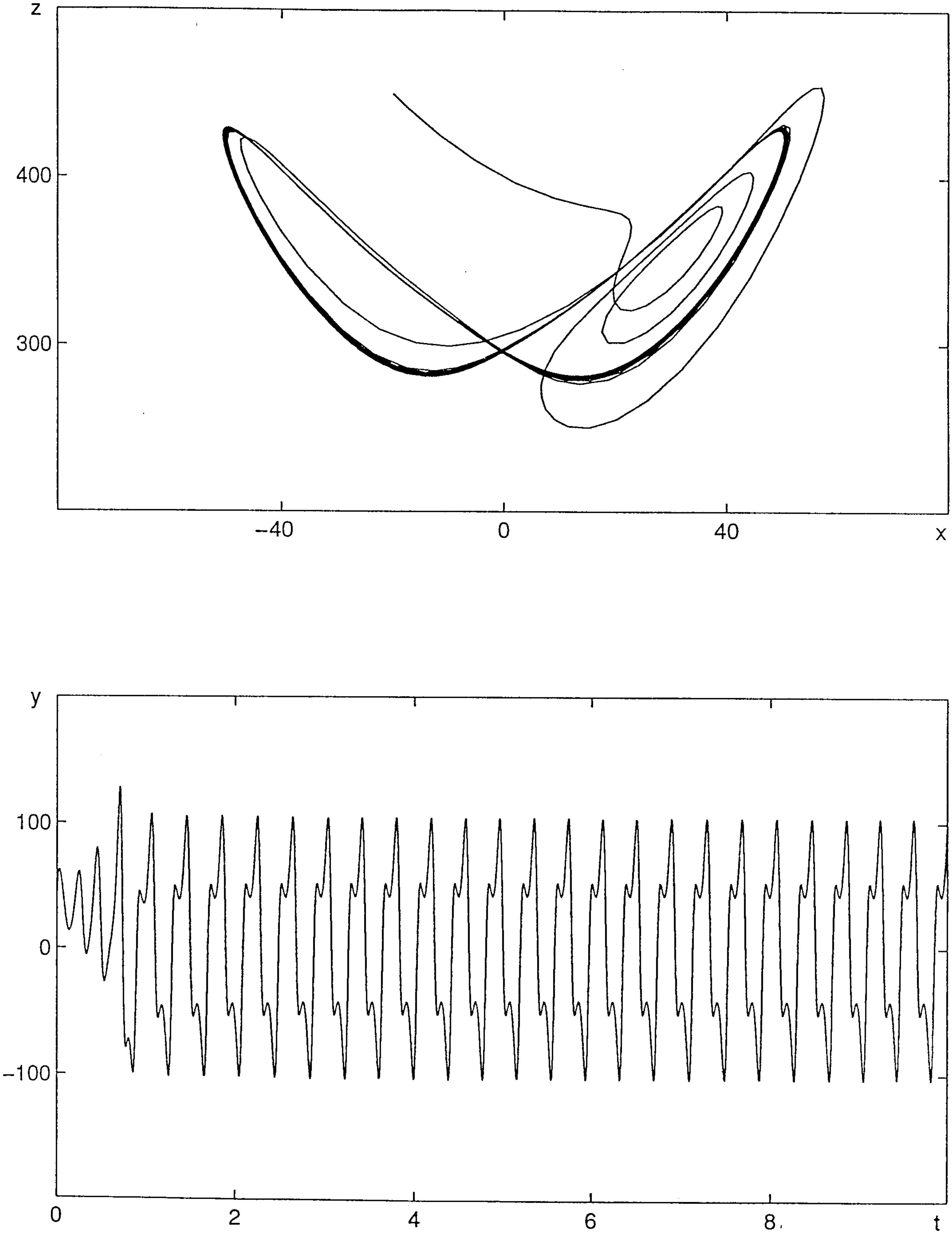
Comments:

The entire bifurcation diagram indicates that there are many routes to chaos.

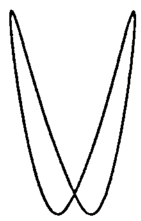
For example,

- period-doubling cascade to chaos
- homoclinic explosion to chaos
- etc.

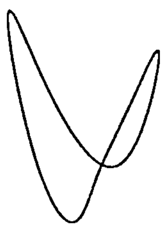
Limit Cycle at r=350



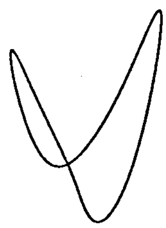
Limit Cycles for $214 < r < \infty$ (the (x, z) projections)

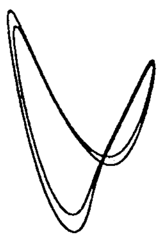


r=350

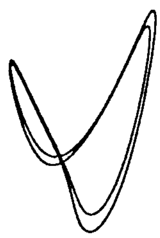


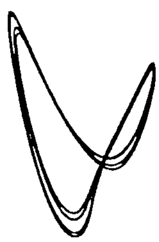
r=250



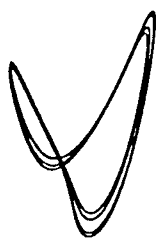


r=220

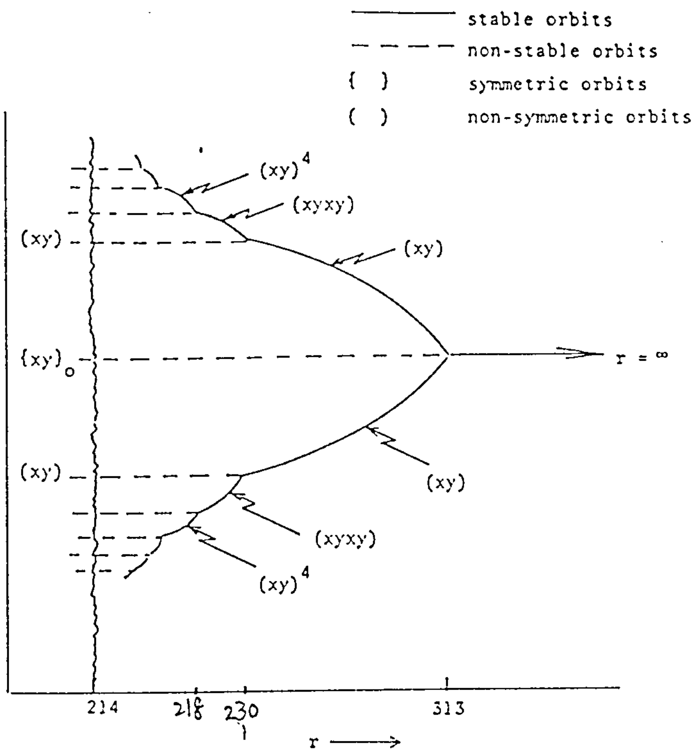


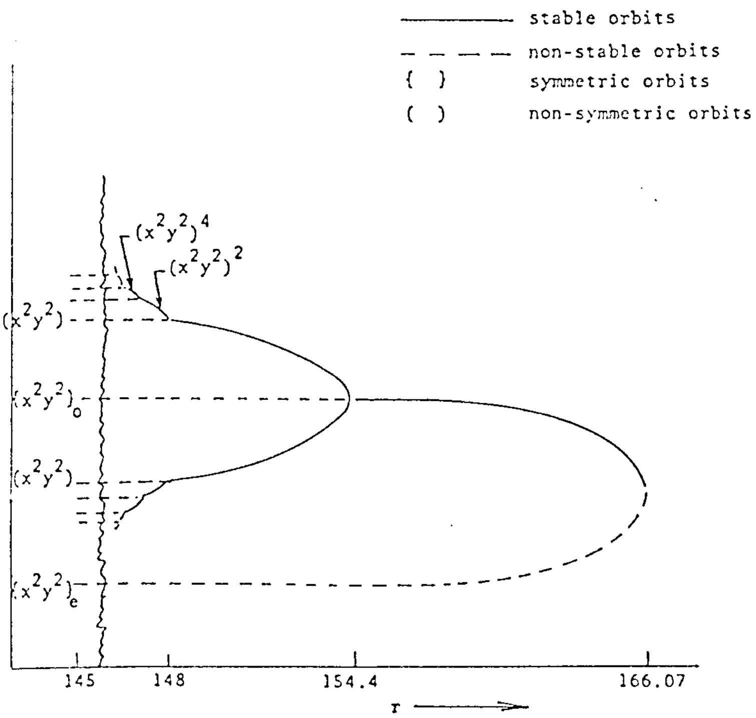


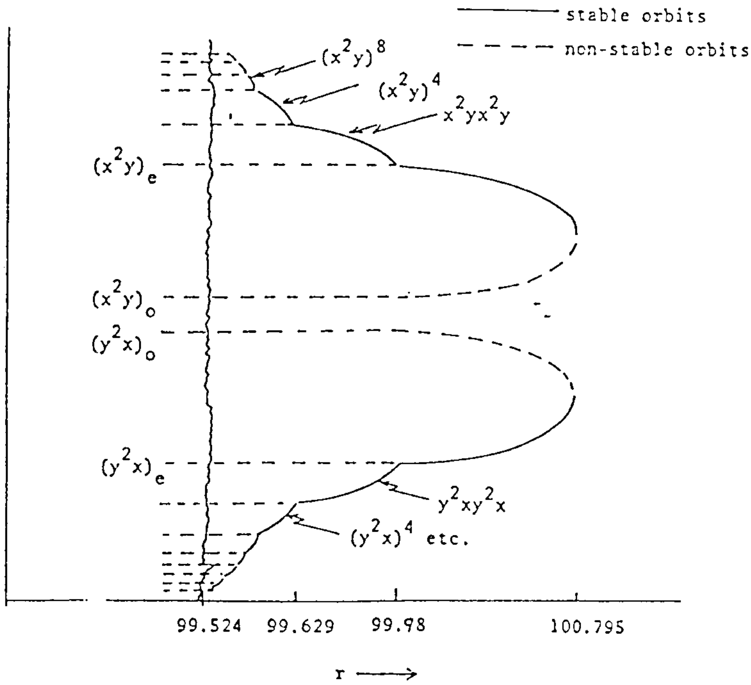
r=216



Bifurcations in the three largest periodic windows







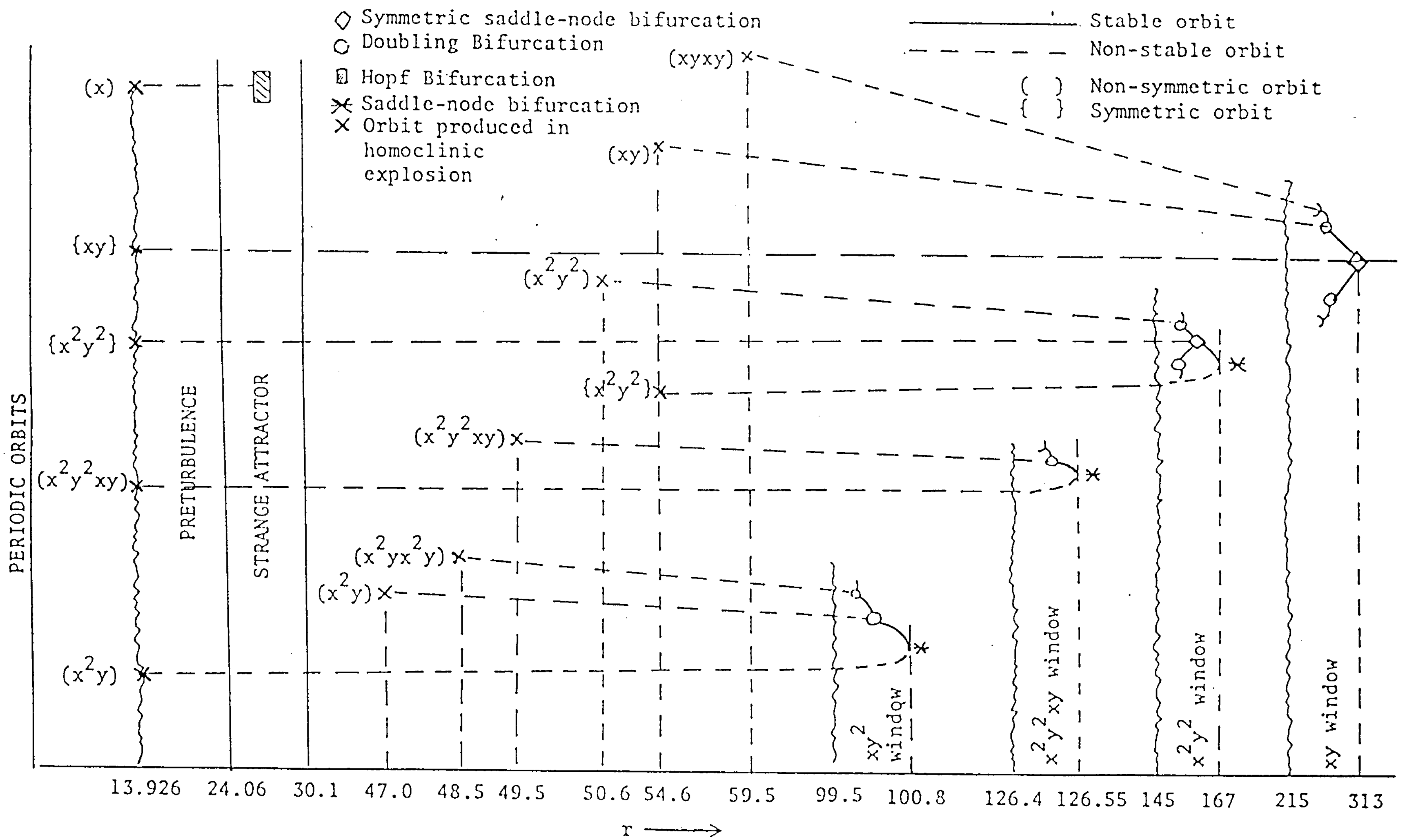


Figure 5.12. Bifurcation diagram showing a few of the shorter periodic orbits.

Computer demonstration on Lorenz equations: $(\sigma = 10, \quad b = \frac{8}{3})$

1. Chaos for Lorenz's parameter: $r = 28.$

   Lorenz map

2. Transient chaos: $13.926 < r < 24.06$

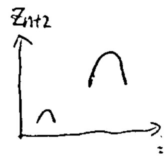
   Take $r = 22$, i.c $(24, 5, 10)$ $t : [0, 70]$

3. Coexistence of two attractors (one strange and the other not): $24.06 < r < 24.74$

   Take $r = 24.5$, i.c. $(7.9, 7.9, 25)$: fixed pt.
   $(1, 0, 0)$: strange attractor

4. large-r behavior: $r = 350.$

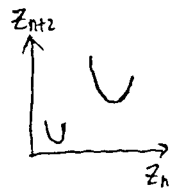
5. Period-doubling bifurcation window r: $[215, 313]$



   $r = 350, \quad 250, \quad 220, \quad 216, \; \bullet\bullet\bullet\bullet$

   Chaos: $r = 210$, Lorenz map $z_{n+2} = f(z_n)$, $\{(z_n, z_{n+2})\}$

6. Another period-doubling window r: $[145, 167]$



   $r =$

   Chaos: $r = 144$, Lorenz map $z_{n+2} = f(z_n)$, $\{(z_n, z_{n+2})\}$