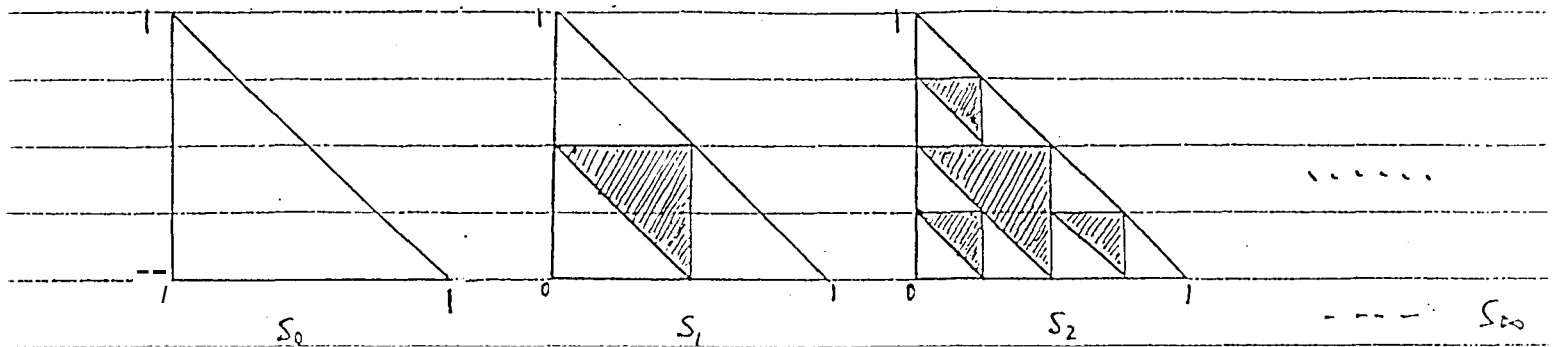


3. Sierpinski gasket

In this section, we discuss a two-dimensional counterpart of the Cantor set - the Sierpinski gasket. It is constructed as follows.



We start with a bilateral triangle whose side length is 1. At first stage, we cut out the open middle quarter which is another bilateral right triangle. Next round, we cut out the open middle quarters of the three remaining smaller right triangles, and so on. This process is repeated infinite times, and what is left in the original right triangle S_0 is called the Sierpinski gasket. This construction is similar to that of the Cantor set, except now it is in two dimensions.

Clearly the Sierpinski gasket is self-similar. It includes infinite line segments, and much more. Its total area is zero.

To determine exactly what points are in this gasket, we adopt the binary number system. To make the representation of a number unique, we forbid infinite consecutive 1s in the digits. Thus, 0.01 is illegal, while 0.1 is okay.

In binary system,

the Sierpinski gasket = $\{(x,y) : 0 \leq x,y \leq 1, \text{ and } x \text{ and } y \text{ do not have 1 simultaneously in a binary place followed by not-all-zero digits}\}$

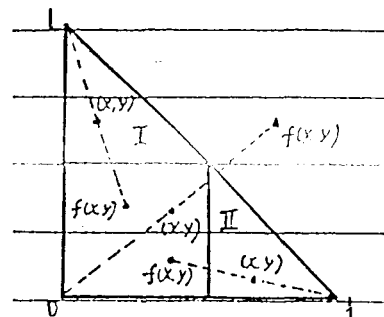
for example, (0.1001,0.0100), (0.1001,0.0101), etc are in the gasket
 (0.1001,0.1100), (0.1001,0.01011), etc are not in it.

Why is it so?

This is not apparent from the construction of the gasket, but is obvious if we associate the gasket with an appropriate map. This is one of the reasons we study the dynamics on this gasket.

Consider the two-dimensional map with

$$f(x,y) = \begin{cases} (2x, 2y - 1), & \text{if } (x,y) \text{ is in or on triangle I} \\ (2x - 1, 2y), & \text{if } (x,y) \text{ is in or on triangle II} \\ (2x, 2y), & \text{otherwise,} \end{cases}$$

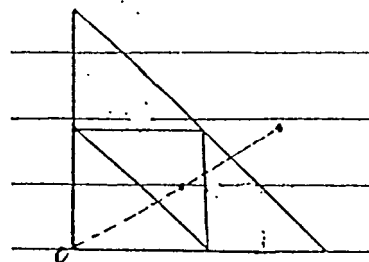


Geometrically speaking, if (x,y) is in triangle I, then draw a straight line from the vertex $(0,1)$ to this point (x,y) , and extend along it the same distance, the point you arrive at is $f(x,y)$; similarly, if (x,y) is in triangle II, you draw a straight line from the vertex $(1,0)$ to (x,y) and extend along it the same distance and reach the point $f(x,y)$; if (x,y) is not in either of these two triangles, then you double the distance from (x,y) to $(0,0)$ and then get $f(x,y)$.

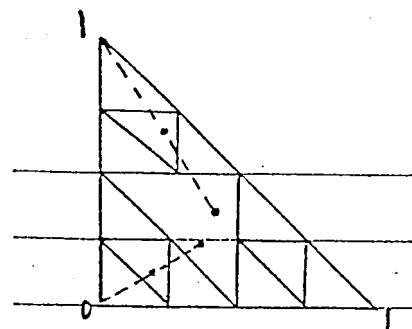
Now we study this map in detail. We are concerned with the question: for a given initial point (x_0,y_0) , what will happen to the orbit $\{f^n(x_0,y_0)\}$ as $n \rightarrow \infty$?

(1) If (x_0,y_0) is outside the original triangle $(0,0), (0,1), (1,0)$, $f^n(x_0,y_0) \rightarrow \infty$.

(2) If (x_0,y_0) is inside the middle triangle, then $f(x_0,y_0)$ lies outside of the original triangle $\Rightarrow f^n(x_0,y_0) \rightarrow \infty$.



(3) If (x_0,y_0) is inside one of the three small triangles, then $f(x_0,y_0)$ lies inside the middle triangle. $\Rightarrow f^2(x_0,y_0)$ lies outside the original triangle $\Rightarrow f^n(x_0,y_0) \rightarrow \infty$, and so on.



Now it is clear that the set

$$\{(x_0,y_0) : f^n(x_0,y_0) \rightarrow \infty\} = \text{the Sierpinski gasket}$$

We have not yet answered the question: why

the Sierpinski gasket = $\{(xy) = 0 \leq x,y \leq 1, \text{ and } x \text{ and } y \text{ do not have } 1 \text{ simultaneously in a binary place followed by not-all-zero digits}\}$?

To do so, let's first simplify the map in binary numbers.

Suppose (x,y) is in the original triangle $(0,0), (0,1)$ and $(1,0)$, and

$$x = 0.a_1a_2a_3 \dots, y = 0.b_1b_2b_3 \dots, \text{ (in binary system)}$$

$$\text{then } f(0.a_1a_2a_3 \dots, 0.b_1b_2b_3 \dots) = (0.a_2a_3a_4 \dots, 0.b_2b_3b_4 \dots)$$

Proof: (1) When (x,y) is in triangle I,

- $$x = 0.0a_2a_3 \dots, y = 0.1b_2b_3 \dots,$$
- $$f(x,y) = (2x, 2y - 1) = (0.a_2a_3a_4 \dots, 0.b_2b_3b_4 \dots),$$
- (2) When (x,y) is in triangle II,
- $$x = 0.1a_2a_3 \dots, y = 0.0b_2b_3 \dots,$$
- $$f(x,y) = (2x - 1, 2y) = (0.a_2a_3a_4 \dots, 0.b_2b_3b_4 \dots)$$
- (3) When (x,y) is in the rest of the original triangle,
- $$x = 0.0a_2a_3 \dots *, y = 0.0b_2b_3 \dots,$$
- $$f(x,y) = (2x, 2y) = (0.a_2a_3a_4 \dots, 0.b_2b_3b_4 \dots). \#$$

Now let us examine what points in the original triangle are not in the gasket.

- (1) If $x = 0.1a_2a_3 \dots, y = 0.1b_2b_3 \dots$ and a_i, b_i are not all zero, then (x,y) is outside the original triangle, thus not in the gasket.
- (2) If $x = 0.01a_3a_4 \dots, y = 0.11b_3b_4 \dots$, or $x = 0.11a_3a_4 \dots, y = 0.01b_3b_4 \dots$, where a_i, b_i are not all zero $f(x,y) = (0.1a_3a_4 \dots, 0.1b_3b_4 \dots)$ which is outside of the original triangle. $\Rightarrow (x,y)$ is not in the gasket.

Similar arguments show that if x and y are 1 simultaneously in a binary place followed by not-all-zero digits, then (x,y) would be in the gasket since $f^n(x,y) \rightarrow \infty$.

Here we introduce a definition. Notice that for $S =$ Sierpinski gasket,

$$f(S) \subseteq S.$$

In other words, the images of S under f still remain in S . We called S an invariant set of the map f . In general,

Definition.

For a map f and a set U , if $f(U) \subseteq U$, then U is called an invariant set of the map f . Notice that if U is an invariant set off; and $x \in U$, then the orbit of x will remain in U forever. Example: the Cantor set is an invariant set of the tent map.

By now, we know that if (x,y) is in the Sierpinski gasket, then $\{f^n(x,y)\}$ would remain in it and be bounded. The next question is: would the orbit be periodic, preperiodic or aperiodic? This is not a hard question from what we know.

Fact If (x,y) is in the Sierpinski gasket and $x = 0.a_1a_2 \dots, y = 0.b_1b_2 \dots$ in binary representation, then

- (1) The point (x,y) is p- eriodic if and only if the digits in x and y are both repeating with period p , i.e. $x = 0.\overline{a_1a_2 \dots a_p}, y = 0.\overline{b_1b_2 \dots b_p}$.
- (2) The point (x,y) is preperiodic if and only if the digits in x and y eventually are both repeating with the same period.
- (3) There are countably infinite periodic points of arbitrary period and countably infinite preperiodic points in the Sierpinski gasket.
- (4) There are uncountably infinite aperiodic points in the gasket.
- (5) The periodic points are dense in the gasket.
- (6) There exists a dense orbit in the gasket.
- (7) The map we constructed depends sensitively on initial conditions everywhere in the gasket.
- (8) The map is chaotic in the gasket.

All of the above statements are easy to verify and will be omitted here.

- (9) The preimages of any point in the gasket are dense in the gasket.
 i.e. for any (x_0, y_0) in the gasket, $\{(x, y) : f^k(x, y) = (x_0, y_0) \text{ for some } k\}$ is dense in the gasket.

Proof: Suppose $x_0 = 0.a_1a_2a_3 \dots$,
 $y_0 = 0.b_1b_2b_3 \dots$,
 then the preimages of (x_0, y_0) are

$$\begin{cases} x = 0.c_k c_{k-1} \dots c_1 a_1 a_2 a_3 \dots \\ y = 0.d_k d_{k-1} \dots d_1 b_1 b_2 b_3 \dots \end{cases}$$

where c_i and $d_i (i = 1, 2, \dots, \text{or } k)$ are not both equal to 1.

Clearly such (x, y) points are dense in the gasket, if one recalls the number structure of the gasket.

For example, if we choose a point $(x_0, y_0) = (0, 0)$. Let us check that its preimages are dense in the gasket.

- (1) For $(0.\overline{10}, 0.\overline{01})$ in the gasket, the preimages of $(0, 0)$ in the form $(0.\underbrace{101010 \dots 10}_{2k}, 0.\underbrace{010101 \dots 01}_{2k})$ can be arbitrarily close to $(0.\overline{10}, 0.\overline{01})$.

- (2) For $(0.1, 0.1)$ in the gasket, the preimages of $(0, 0)$ in the form $(0.1, 0.\underbrace{01111 \dots 1}_k)$ can be arbitrarily close to $(0.1, 0.1)$, and so on.

When one traces the preimages of a point (x_0, y_0) , the preimages he gets will bear less and less relationship to the starting point (x_0, y_0) every time he traces a step back. If he traces infinite steps back, the preimages will be totally independent of the point (x_0, y_0) and can be almost arbitrary. This is somewhat similar to the ancestor tracing in the human race.

Note: The preimages of any point in the Cantor set under the tent map are also dense in C. The proof is similar to the one above.

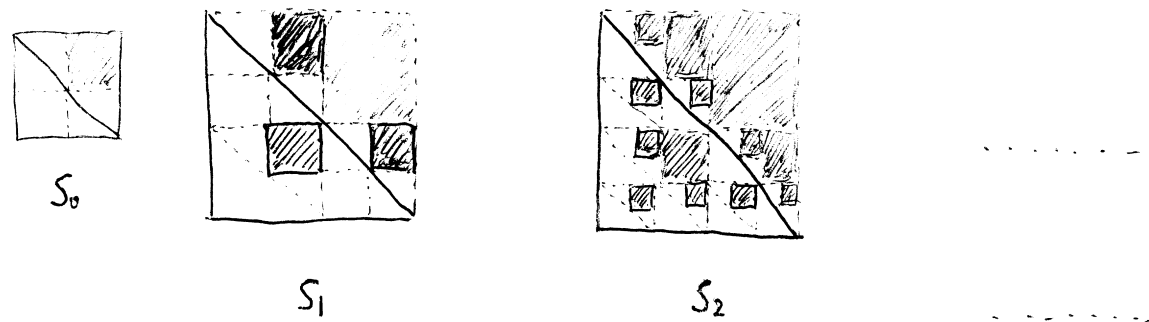
Comment: Property 9 automatically implies Property 7 (sensitive dependence on initial conditions).

To prove the description of the Sierpinski gasket, which is

$$S = \{(x, y) : 0 \leq x, y \leq 1, \text{ and } x \text{ and } y \text{ do not have } 1 \text{ simultaneously in a binary place followed by not-all-zero digits}\}$$

or $\bar{S} = \{(x, y) : 0 \leq x, y \leq 1, \text{ and } x \text{ and } y \text{ have } 1 \text{ simultaneously in a binary place followed by not-all-zero digits}\}$,

by geometric means, we modify the construction of the gasket as follows:



At the first stage: we remove the box $\begin{cases} x=0.1\dots \\ y=0.1\dots \end{cases} \Rightarrow S_0$

Second stage: we remove the $\begin{matrix} 3 \\ \downarrow \end{matrix}$ boxes $\begin{cases} x=0.01\dots, 0.11\dots, 0.01\dots \\ y=0.01\dots, 0.01\dots, 0.11\dots \end{cases} \Rightarrow S_1$

third stage: we remove the $\begin{matrix} 9 \\ \downarrow \end{matrix}$ boxes

$$\begin{cases} x=0.001\dots, 0.001\dots, 0.001\dots, 0.001\dots, 0.011\dots, 0.011\dots, \\ y=0.001\dots, 0.011\dots, 0.011\dots, 0.111\dots, 0.001\dots, 0.101\dots, \\ 0.101\dots, 0.101\dots, 0.111\dots \\ 0.001\dots, 0.011\dots, 0.001\dots \end{cases} \Rightarrow S_2$$

After infinite stages of removal $\Rightarrow S_\infty =$ the Sierpinski gasket.