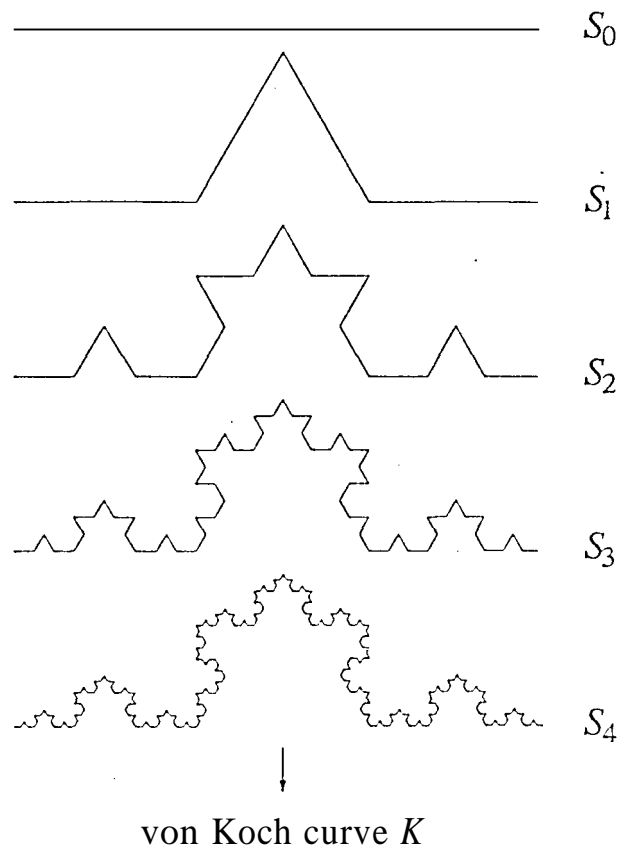


§5. Dimension of fractals.

The objects we studied before, namely the Cantor set, the Sierpinski gasket and the Julia set of Newton's method for $z^3 = 1$, are all examples of fractals. The amazing thing is that the micro-structures of these objects are as rich as their macrostructures. In other words, these objects do not lose their structures (details) when repeatedly magnified. This is strange because in all the objects we have studied before – lines, curves, surfaces, solids, etc, their micro-structures are flat and dull. For instance, a curve in calculus, no matter how "bumpy" it appears to the eye, becomes a straight line upon repeated magnification.

What is the dimension of a fractal? For familiar geometric objects, the answer is clear. For instance, lines and smooth curves are one-dimensional, planes and smooth surfaces are two-dimensional, solids are three-dimensional, and so on. If forced to give a definition, we could say that the dimension is the minimum number of coordinates needed to describe every point in the object. For instance, a smooth curve is one-dimensional because every point on it is determined by one number, the arc length from some fixed reference point on the curve. But when we try to apply this definition to fractals, we quickly run into trouble.

To make this point clear, we consider a new fractal – the von Koch curve, defined recursively in Fig 1. We start with a line segment S_0 . To generate S_1 , we delete the middle third of S_0 and replace it with the other two sides of an equilateral triangle. Subsequent stages are generated recursively by the same rule: S_n is obtained by replacing the middle third of each line segment in S_{n-1} by the other two sides of an equilateral triangle. The limiting set $K = S_\infty$ is the von Koch curve. This curve is self-similar, similar to the fractals studied before.



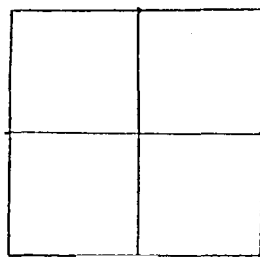
What is the dimension of the von Koch curve? Since it is a curve, you might be tempted to say it is one-dimensional. But it has infinite arc length (the reason is that $L_{n+1} = \frac{4}{3}L_n$, where L_n is the arc length of S_n . Thus $L_n = (\frac{4}{3})^n L_0 \rightarrow \infty$ as $n \rightarrow \infty$). Furthermore, the arc length between any two points on K is infinite (for the same reason). Hence points on K are not determined by their arc length from a particular point, because every point is infinitely far from every other!

This suggests that K is more than one-dimensional. But is it two-dimensional? Of course not, because it does not have any "area". So its dimension should be between 1 and 2, whatever that means.

The idea that a set or an object can have a non-integer dimension is very novel. It is motivated primarily by exotic objects which are now called fractals. Many definitions for the dimension of a set have been proposed. We only discuss a couple of them, among which is the Hausdorff dimension which is popular among mathematicians.

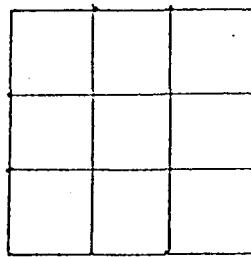
1. Similarity dimension.

Some simple fractals such as the Cantor set, the Sierpinski gasket and the Koch curve are self-similar, i.e. portions of the set are copies of the whole set at a reduced scale. The dimension of such fractals can be defined by extending an elementary observation about classical self-similar sets like line segments, squares or cubes. For instance, consider the square region show below:



$$m = 4$$

$$r = 2$$



$$m = 9$$

$$r = 3$$

m : number of copies

r : scale factor

$$m = r^2$$

If we shrink the square by a factor of 2 in each direction, the original square will consist of 4 scaled-down squares. In general, if we shrink the square by a factor of r , then

$$m = r^2$$

where m is the number of scaled-down copies in the original square.

The reader should note that the exponent 2 in the above formula is no accident. It reflects the two-dimensionality of the square. This connection between dimensions and exponents suggests the following definition.

Definition: Suppose that a self-similar set is composed of m copies of itself scaled down by a factor of r . Then the similarity dimension d is defined by $m = r^d$, or equivalently

$$d = \frac{\ln m}{\ln r}$$

This formula is easy to use, since m and r are usually clear from inspection.

Example 1. The similarity dimension of the Cantor set C .

Solution: here $m = 2$ when $r = 3 \Rightarrow d = \frac{\ln 2}{\ln 3} \approx 0.63$.

Example 2. The similarity dimension of the Sierpinski gasket.

Solution: now $m = 3$ when $r = 2 \Rightarrow d = \frac{\ln 3}{\ln 2} \approx 1.58$.

Example 3. The similarity dimension of the Koch curve.

Solution: when $r = 3$, $m = 4 \Rightarrow d = \frac{\ln 4}{\ln 3} \approx 1.26$.

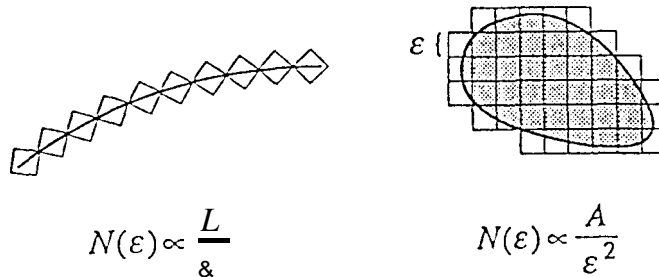
The similarity dimension is easy to handle. But it has many limitations. First, it only applies to self-similar sets. As we know, some fractals, such as the Julia set of Newton's method applied to equation $z^3 = 1$, are not strictly self-similar. Second, this dimension is not topologically flexible. For instance, if the Cantor set is continuously deformed, then its self-similarity can be easily destroyed. But one would still expect the dimension of the deformed Cantor set to be the same as the self-similar one, just like a deformed square should still have dimension 2. The definition of similarity dimension does not meet this expectation.

2. Box dimension

To deal with fractals which are not self-similar, we need more general definitions for dimension. Box dimension is one of them.

Let S be a subset of D -dimensional Euclidean space,
and $N(\epsilon)$ be the minimum number of D -dimensional cubes of side ϵ needed to cover S .

How does $N(\epsilon)$ depend on ϵ , especially when $\epsilon \rightarrow 0$?



Obviously, for a smooth curve of length L , $N(\epsilon) \propto \frac{L}{\epsilon}$;

for a planar region of area A , $N(\epsilon) \propto \frac{A}{\epsilon^2}$.

Notice that in both cases, $N(\epsilon) \propto \frac{1}{\epsilon^d}$, where d is the familiar dimension of the set. We extend this observation to a general set and propose the following definition.

Definition: The box dimension of a set S is defined as

$$d = \lim_{\epsilon \rightarrow 0} \frac{\ln N(\epsilon)}{\ln(1/\epsilon)}, \text{ if the limit exists.}$$

Note: the above limit does not depend on the actual size of the set S . For instance, if S is enlarged by a factor of 3, then it takes $3N(\varepsilon)$ ε -cubes to cover it. But the above limit for d still remains the same. This makes sense because a line segment of length 1 and another one of length 3 should be both one-dimensional.

Example: Find the box dimension of the Cantor set.

Solution: Recall that the Cantor set is covered by each of the sets S_n used in its construction (refer to previous notes on the Cantor set). Each S_n consists of 2^n intervals of length 3^{-n} , so if $3^{-n} \leq \varepsilon < 3^{1-n}$, then the minimum number of such coverings $N(\varepsilon)$ is 2^n .

Hence
$$\frac{\ln 2^n}{\ln 3^n} \leq \frac{\ln N(\varepsilon)}{\ln(1/\varepsilon)} < \frac{\ln 2^n}{\ln 3^{n-1}} \Rightarrow \frac{\ln 2}{\ln 3} \leq \frac{\ln N(\varepsilon)}{\ln(1/\varepsilon)} < \frac{n \ln 2}{(n-1) \ln 3}.$$

As $n \rightarrow \infty, \varepsilon \rightarrow 0$, thus

$$d = \lim_{\varepsilon \rightarrow 0} \frac{\ln N(\varepsilon)}{\ln(1/\varepsilon)} = \lim_{n \rightarrow \infty} \frac{\ln N(\varepsilon)}{\ln(1/\varepsilon)} = \frac{\ln 2}{\ln 3}.$$

Notice that for the Cantor set, the box dimension and the similarity dimension are the same. This is no accident. In general, we have the following result:

Theorem: If S is a self-similar set, then its box and similarity dimensions are the same.

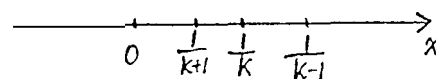
This result can be proved by a similar idea as used above for the Cantor set.

Box dimension is an extension of the similarity dimension. It can be applied to any set. But it is not fool-proof. Its value is not always what it should be. For instance, the set of rational numbers between 0 and 1 can be shown to have box dimension 1, even though the set has countably many points. More generally, the box dimension of the closure of a set is the same as for the set itself. Thus we recommend that box dimension be used only for closed sets.

Even if the box dimension is restricted to closed sets, difficulties still remain.

Example: $F = \{0, 1, \frac{1}{2}, \frac{1}{3}, \dots\}$ is a compact set with box dimension $d = \frac{1}{2}$.

Proof: If $\frac{1}{K(K+1)} \leq \varepsilon < \frac{1}{K(K-1)}$,



i.e. $\frac{1}{K} - \frac{1}{K+1} \leq \varepsilon < \frac{1}{K-1} - \frac{1}{K}$,

then an ε -interval can cover at most one of the points $(1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{K})$.

Thus at least K ε -intervals are needed to cover F , i.e. $N(\varepsilon) \geq K$.

To cover the other points $\{\frac{1}{K+1}, \frac{1}{K+2}, \dots\}$, at most $\frac{1}{K+1} / \varepsilon \leq K$ more ε -intervals are needed, i.e. $N(\varepsilon) \leq 2K$.

Hence $K \leq N(\epsilon) \leq 2K, \Rightarrow \frac{\ln K}{\ln K(K+1)} \leq \frac{\ln N(\epsilon)}{\ln(1/\epsilon)} \leq \frac{\ln 2K}{\ln K(K-1)}$

As $K \rightarrow \infty, \epsilon \rightarrow 0$, and

$$d = \lim_{\epsilon \rightarrow 0} \frac{\ln N(\epsilon)}{\ln(1/\epsilon)} = \lim_{K \rightarrow \infty} \frac{\ln N(\epsilon)}{\ln(1/\epsilon)} = \frac{1}{2}.$$

A good dimension definition would ensure that the dimension of a countable set is zero. The box dimension does not.

3. Hausdorff dimension

An improved version of the box dimension is the Hausdorff dimension. This dimension is the most commonly used by mathematicians, but its definition is not short and easy. The main difference between Hausdorff and box dimensions is that the Hausdorff dimension uses coverings by small sets of varying sizes, not just boxes of fixed size ϵ .

First we introduce some notations.

If $x = (x_1, x_2, \dots, x_n) \in R^n$, define $|x| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$, the Euclidean norm of x .

If A is a nonempty subset of R^n , we define the diameter of A as

$$|A| = \sup\{|x - y| : x, y \in A\}.$$

If $\{A_i\}$ is a countable (or finite) collection of sets of diameter at most δ that cover F , i.e. $F \subset \bigcup_{i=1}^{\infty} A_i$, with $0 < |A_i| \leq \delta$ for each i , we say that $\{A_i\}$ is a δ -cover of F .

Hausdorff measure

Suppose that F is a subset of R^n and s is a non-negative number. For any $\delta > 0$ we define

$$H_{\delta}^s(F) = \inf \left\{ \sum_{i=1}^{\infty} |A_i|^s : \{A_i\} \text{ is a } \delta\text{-cover of } F \right\}.$$

As δ decreases, the class of permissible covers of F in the above definition is reduced. Therefore the infimum $H_{\delta}^s(F)$ increases, and so approaches a limit as $\delta \rightarrow 0$. We write

$$H^s(F) = \lim_{\delta \rightarrow 0} H_{\delta}^s(F).$$

This limit exists for any subset F of R^n , though the limiting value can be (and usually is) 0 or ∞ . We call $H^s(F)$ the s -dimensional Hausdorff measure of F .

Properties of the Hausdorff measure:

1. $H^s(\emptyset) = 0$.
2. If $E \subset F$, then $H^s(E) \leq H^s(F)$;

3. If $\{F_i\}$ is any countable collection of disjoint Borel sets, then $H^s\left(\bigcup_{i=1}^{\infty} F_i\right) = \sum_{i=1}^{\infty} H^s(F_i)$.
4. If $F \subset \mathbb{R}^n$ and $\lambda > 0$, then $H^s(\lambda F) = \lambda^s H^s(F)$, where $\lambda F = \{\lambda x : x \in F\}$, i.e. the set F scaled by a factor λ .

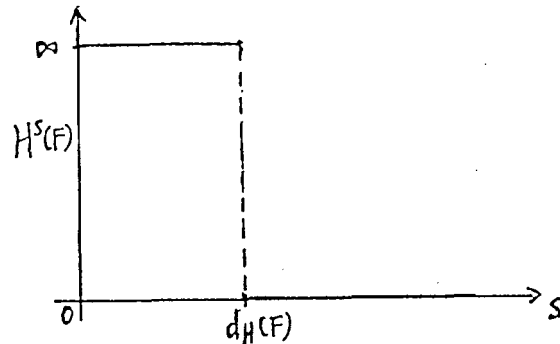
The first three properties indicate that H^s is indeed a measure. Proofs for (1), (2) and (4) are simple, but the one for (3) is more involved.

Hausdorff dimension

Recall that if $t > s > 0$ and $\{A_i\}$ is a δ -cover of F , then we have

$$\sum_i |A_i|^t \leq \delta^{t-s} \cdot \sum_i |A_i|^s.$$

So taking infima, $H_\delta^t(F) \leq \delta^{t-s} \cdot H_\delta^s(F)$. Letting $\delta \rightarrow 0$ we see that if $H^s(F) < \infty$, then $H^t(F) = 0$ for $t > s$. Thus a graph of $H^s(F)$ against s shows that there is a critical value of s at which $H^s(F)$ "jumps" from ∞ to 0. This critical value is called the Hausdorff dimension of F , and is written as $d_H(F)$.



Formally $d_H(F) = \inf\{s : H^s(F) = 0\} = \sup\{s : H^s(F) = \infty\}$.

If $s = d_H(F)$, then $H^s(F)$ may be zero or infinity, or may satisfy $0 < H^s(F) < \infty$.

Example: If F is a flat disk, then we can show that $H^1(F) = w$, $H^2(F) = \frac{\pi}{4} \cdot \text{area}(F)$, $H^3(F) = 0$, $\Rightarrow d_H(F) = 2$.

Properties of Hausdorff dimension:

1. Open sets: if $F \subset \mathbb{R}^n$ is open, then $d_H(F) = n$.
2. If $E \subset F$, then $d_H(E) \leq d_H(F)$.
3. $d_H\left(\bigcup_{i=1}^{\infty} F_i\right) = \sup_{1 \leq i < \infty} \{d_H(F_i)\}$.
4. If F is countable, then $d_H(F) = 0$.
5. For any set F , $d_H(F) \leq d_B(F)$, where $d_B(F)$ is the box dimension of F .
6. For a self-similar set F , $d_H(F) = d_B(F) = d_s(F)$, where $d_s(F)$ is the similarity dimension of F .

7. A set $F \subset \mathbb{R}^n$ with $d_H(F) < 1$ is totally disconnected.

Example 1: The Hausdorff dimension of the rational numbers in $[0,1]$ is zero.

Example 2: If $F = \{0,1, \frac{1}{2}, \frac{1}{3}, \dots\}$, then $d_H(F) = 0$.

Example 3. The Hausdorff dimension of the Cantor set is $\frac{\ln 2}{\ln 3}$.

Note that the Hausdorff dimension in examples 1 and 2 gives us values we expected.

Defining fractals

Although we have seen quite a few fractals such as the Cantor set, the Koch curve and the Julia set of Newton's method applied to equation $z^3 = 1$, we have not given a mathematical definition for it. The most commonly used definition is as follows (it was given by B. Mandelbrot).

Definition: A fractal is a set whose Hausdorff dimension is greater than its topological dimension.

Examples: The topological dimension of the Koch curve is 1, while its Hausdorff dimension is $\frac{2 \ln 2}{\ln 3} \approx 1.26$, thus it is a fractal.

The Cantor set and the Julia set are all fractals according to this definition.

On the other hand, the set of rational numbers in $[0,1]$ and the set $\{0,1, \frac{1}{2}, \frac{1}{3}, \dots\}$ are not fractals. This makes sense. They are too simple to be called a fractal.