

§6. The logistic map, period-doubling to chaos

One of the simplest, yet most intriguing maps is the logistic map. The wealth of information gathered on this map in the last twenty years revealed its very complicated dynamics and universality. We discuss it in this section.

The logistic map looks very innocent:

$$f(x) = rx(1-x), \quad x \in [0,1], \quad r > 0.$$

Question: for given $x_0 \in [0,1]$, what happens to $f^n(x)$ as $n \rightarrow \infty$?

1. $0 < r < 1$:

In this case, $f(x) = rx(1-x) = x$ gives us a single fixed point $x = 0$.

Since $f'(0) = r \Rightarrow x = 0$ is attracting.

Actually, we can show that it is also globally attracting, i.e.

$$f^n(x_0) \rightarrow 0 \text{ as } n \rightarrow \infty \text{ for any } x_0 \in [0, 1].$$

Proof: $|f(x)| < r|x| \Rightarrow |f^2(x)| < r|f(x)| < r^2|x|, \dots$

$$\Rightarrow |f^n(x)| < r^n|x| \Rightarrow |f^n(x)| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

2. $1 < r < 3$:

In this case, solving $f(x) = x$ gives us two fixed points $x = 0$ and $x = 1 - \frac{1}{r}$.

$$f'(0) = r > 1 \Rightarrow x = 0 \text{ is repelling}$$

$$|f'(1 - \frac{1}{r})| = |2 - r| < 1 \Rightarrow x = 1 - \frac{1}{r} \text{ is attracting.}$$

Are there periodic points in this case? No.

In fact, we can show that,

$$f^n(x_0) \rightarrow 1 - \frac{1}{r} \text{ as } n \rightarrow \infty \text{ for any } x_0 \in (0,1).$$

Sketch of the proof:

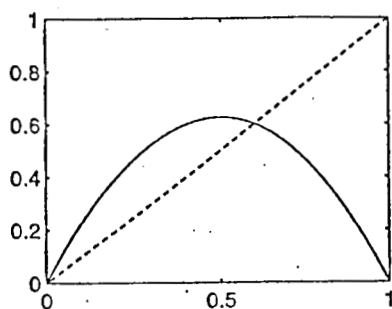
$$|f(x)| = |r - 2rx| < 1 \Leftrightarrow |x - \frac{1}{2}| < \frac{1}{2r} \Leftrightarrow x \in (\frac{1}{2} - \frac{1}{2r}, \frac{1}{2} + \frac{1}{2r}).$$

Note that $1 - \frac{1}{r} \in (\frac{1}{2} - \frac{1}{2r}, \frac{1}{2} + \frac{1}{2r})$ and $|f'(x)| < 1$ in this interval

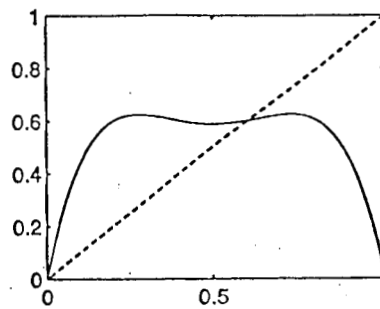
$$\Rightarrow \text{for any } x_0 \in (\frac{1}{2} - \frac{1}{2r}, \frac{1}{2} + \frac{1}{2r}), \quad f^n(x_0) \rightarrow 1 - \frac{1}{r} \text{ as } n \rightarrow \infty.$$

Then we can **show** that all the preimages of this interval are $(0,1)$. #

Graph of $f(x) = 2.5x(1-x)$



Graph of $f^2(x)$



3. $r > 3$: In this case, both of the two fixed points are repelling. But the periodic points appear.

(a) 2-cycle

Solving $f^2(x) = x$ gives us four roots $x = 0, 1 - \frac{1}{r}, \frac{r+1 \pm \sqrt{(r-3)(r+1)}}{2r}$

The first two are clearly the repelling fixed points. The second two are two-periodic points.

Stability: $(f^2)'(x_0) = f'(x_1)f'(x_0)$ where $x_1 = f(x_0)$.

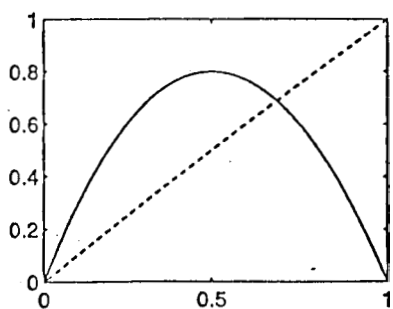
When $x_0 = \frac{r+1 \pm \sqrt{(r-3)(r+1)}}{2r}$, $(f^2)'(x_0) = 4 + 2r - r^2$.

Therefore the 2-cycle is attracting for $|4 + 2r - r^2| < 1$.

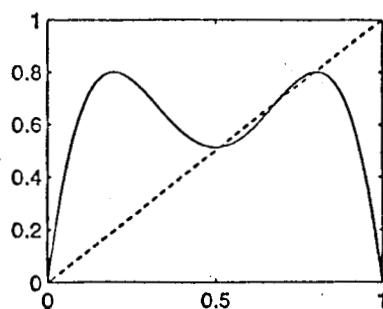
i.e. $3 < r < 1 + \sqrt{6} \approx 3.449\dots$

When $r > 1 + \sqrt{6}$, it is repelling.

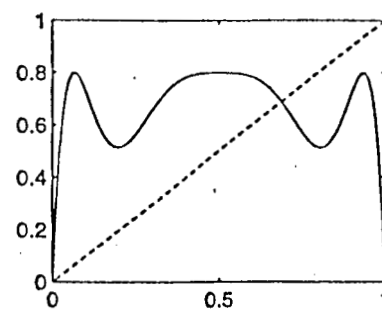
Graph of $f(x) = 3.2x(1-x)$



Graph of $f^2(x)$



Graph of $f^3(x)$



When $3 < r < 1 + \sqrt{6}$, are there other periodic orbits besides the 2-cycle? No. Actually it can be shown that, except a countable number of points in $[0, 1]$ (which are 0, 1 and the preimages of the repelling fixed point $1 - \frac{1}{r}$), every point is eventually attracted to this 2-cycle. Numerically, one usually finds that every point in $[0, 1]$ is attracted to this 2-cycle because both 0 and $1 - \frac{1}{r}$ are repelling.

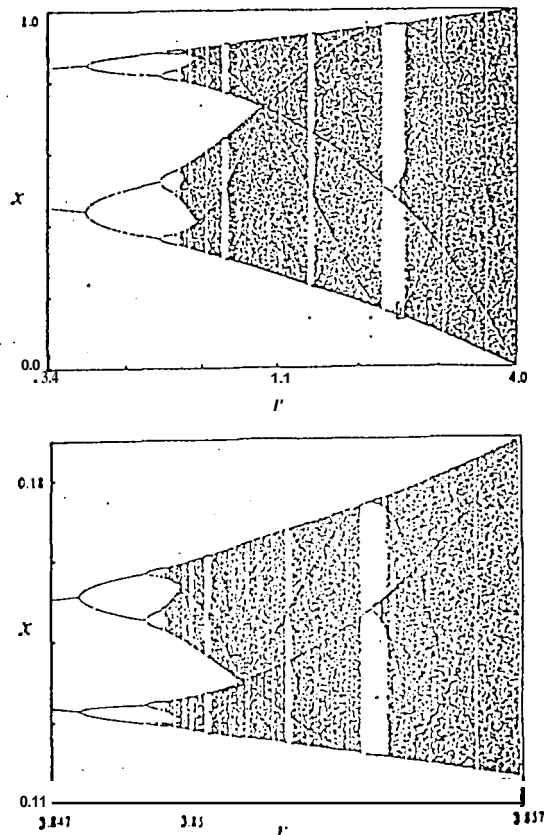
What happens when $r > 1 + \sqrt{6}$?

In this case, the 2-cycle is repelling, but a 4-cycle appears. Clearly the analysis gets more and more awkward. Hence we switch to numerical exploration. But keep in mind that numerics can find attractors but miss the unstable structures. The following is what happens when $4 > r > 1 + \sqrt{6} \approx 3.449$.

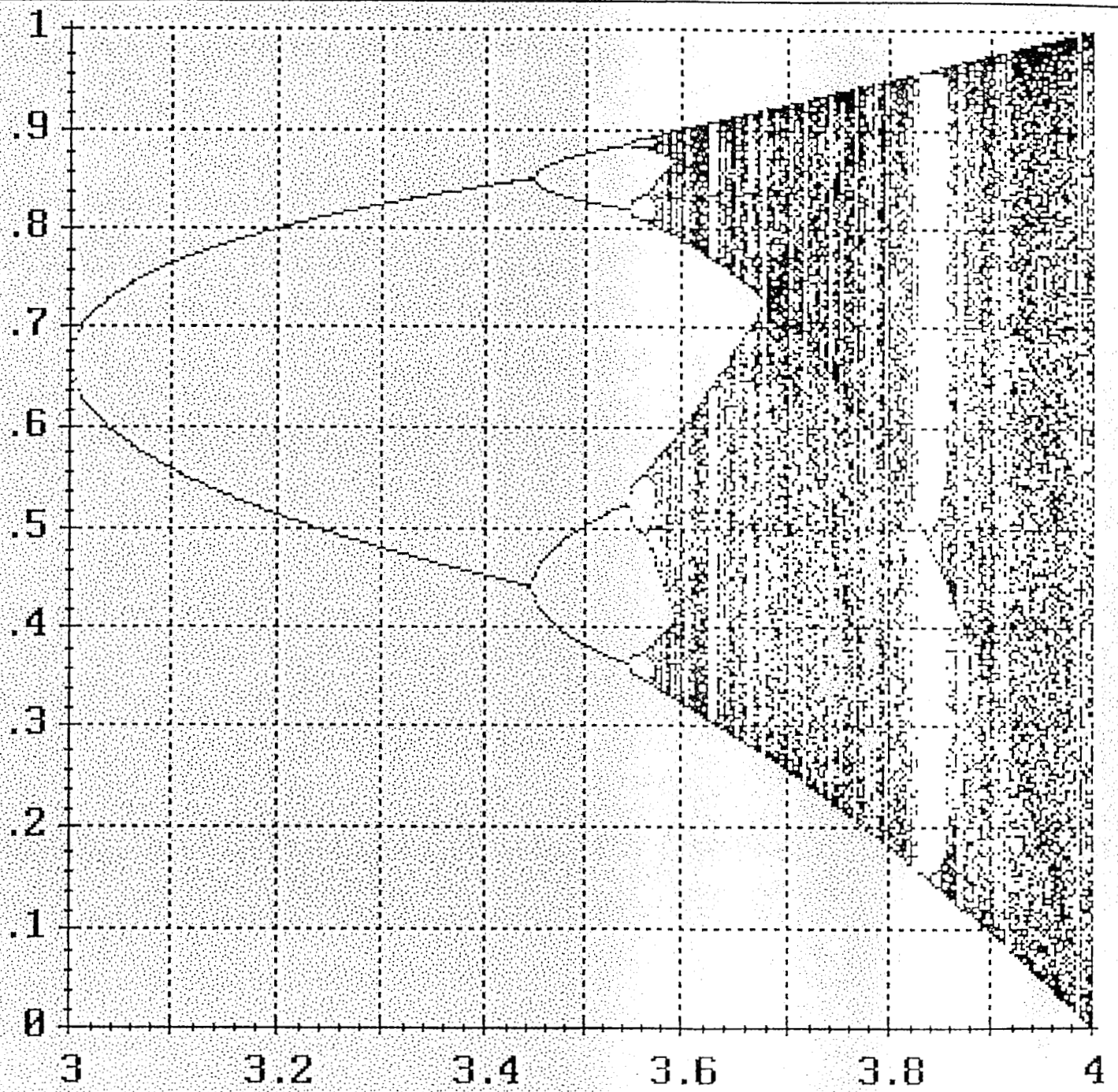
- 3.449... < r < 3.54409...: there is an attracting 4-cycle. Numerically every point is eventually attracted to it.
- 3.54409... < r < 3.5644...: there is an attracting 8-cycle which numerically attracts every point in $(0, 1)$.
- 3.5644.. < r < 3.568759.: there is an attracting 16-cycle.

This is the so-called period-doubling bifurcation. Note that the successive bifurcations come faster and faster. Ultimately the intervals of 2^n -cycles converge to a point $r_\infty \approx 3.569946...$ as $n \rightarrow \infty$.

What happens if $r > r_\infty$? Here, the orbit $\{f^n(x_0)\}$ is chaotic for most r values. But periodic windows mysteriously appear here and there. The largest window is near $(3.8284, 3.8415)$ which is 3-periodic. The complete orbit diagram, which is the plot of the map's attractor as a function of r , is given below. This amazing diagram is as beautiful as it is mysterious. If you look at it more closely, you will find that, lying just above the periodic windows in the chaotic region are small copies of the whole orbit diagram. Thus this picture has fine structures at arbitrarily small scales.



$$F(x) = ax(1-x)$$

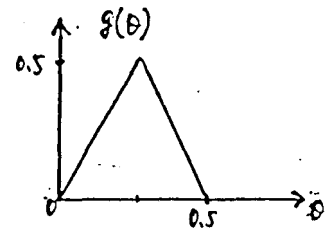


A special case of chaos: $r = 4$

When $r = 4$, the logistic map is $f(x) = 4x(1-x)$, $x_{n+1} = 4x_n(1-x_n)$.

Under the variable transform: $x = \sin^2 \pi \theta$, $(0 \leq \theta \leq \frac{1}{2})$
 the map becomes: $\sin^2 \pi \theta_{n+1} = 4 \sin^2 \pi \theta_n (1 - \sin^2 \pi \theta_n) = \sin^2 2\pi \theta_n$,

or $\theta_{n+1} = g(\theta_n) = \begin{cases} 2\theta_n, & \text{if } 0 \leq \theta_n \leq \frac{1}{4}, \\ \theta_n, & \text{if } \frac{1}{4} \leq \theta_n \leq \frac{1}{2}. \end{cases}$

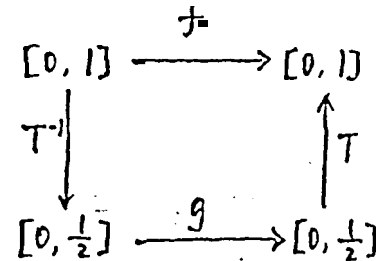


This is a simple tent map, where dynamics is well-known. In particular, g has periodic points of any period in the interval $[0, 0.5]$, and its dynamics on it is chaotic. If we denote $T(\theta) = \sin^2 \pi \theta$, then it is easy to show that

$$f = T \circ g \circ T^{-1}$$

(topological conjugacy).

Thus f is also chaotic on $[0, 1]$

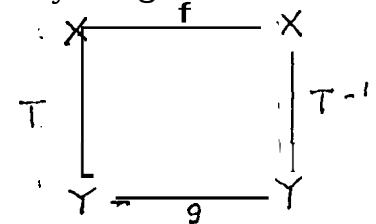


Note: the complete definition of topological conjugacy is as follows.

Definition: Let X and Y be two subsets of the real line and let f and g be functions, $f : X \rightarrow X$ and $g : Y \rightarrow Y$. Then f and g are said to be topologically conjugate provided f and g are continuous, and there is a homeomorphism $T : X \rightarrow Y$ such that

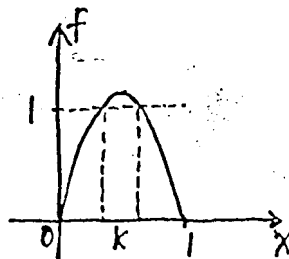
$$T \circ f(x) = g \circ T(x), \quad \text{or } f = T^{-1} \circ g \circ T$$

holds for all $x \in X$



Note: A function $T : X \rightarrow Y$ is said to be a homeomorphism provided T is continuous, one-to-one and onto, and the inverse T^{-1} is also continuous.

What happens when $r > 4$?



In this case, $f(x) > 1$ for some values of x in the middle of the interval $[0, 1]$. Suppose $f(x) > 1$ when $x \in K$ (refer to the graph). Then for $x \in K$, $f(x) > 1$, $f^2(x) < 0$, \dots , $f^n(x) \rightarrow -\infty$ as $n \rightarrow \infty$. It is easy to see that all the preimages of K also escape to negative infinity. Then what is the set

$$S = \{x : f^n(x) \rightarrow -\infty\} ?$$

It is a set similar to the Cantor set:

- (1) It is totally disconnected. **All** points in S are separated from each other;
- (2) On the other hand, it contains no "isolated points".

We call such a set a topological Cantor set. Hence, when $r > 4$, the invariant set of f is a topological Cantor set.

Sarkovskii's Theorem

Next we discuss a remarkable theorem on the dynamics of real maps (such as the logistic map). It is amazing for its lack of hypotheses and strong conclusion. This theorem was given first by Alexander Sarkovskii (also Charkovsky) in 1964 (Ukr. Mat. J. 16, 61-71). It was rediscovered by Li, T.Y. and Yorke, J.A. in 1975 (American Mathematical Monthly 82, 985-992).

Theorem 1. Let $f: R \rightarrow R$ be continuous. Suppose f has a periodic point of period three, then it has periodic points of all other periods.

Proof: The proof will depend on two elementary observations:

- (1) If I and J are closed intervals with $I \subset J$ and $f(I) \supset J$, then f has a fixed point in I ; (see Fig 1).
- (2) If I and J are closed intervals and $f(I) \supset J$, then there exists at least one subinterval K of I which is mapped onto J (see Fig 2).

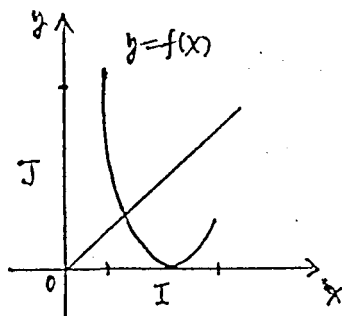


Figure 1

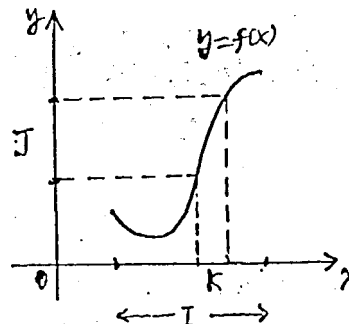


Figure 2

These observations are clear from the above diagrams and are consequences of the intermediate value theorem of continuous functions.

To prove the theorem, let $a, b, c \in R$ and suppose $f(a) = b, f(b) = c$ and $f(c) = a$. We assume that $a < b < c$. The only other possibility $a < c < b$ can be handled similarly.

Let $I_0 = [a, b]$ and $I_1 = [b, c]$.

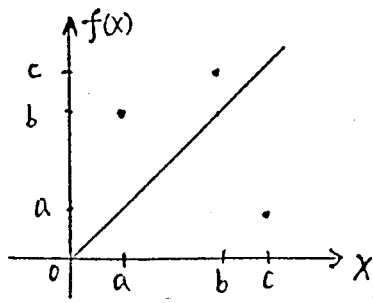


Figure 3

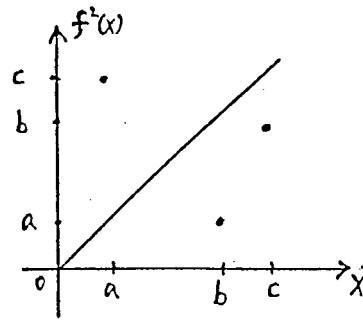


Figure 4

It is clear from Fig 3 that f must have fixed points in (a, b) . Similarly, we see from Fig 4 that f^2 must have fixed points in (a, b) . It is easy to show that at least one of these points must have period two (refer to the arguments below for details). Next we show that f has a periodic point of any prime period $n > 3$. To do this, we define a nested sequence of intervals $A_0, A_1, A_2, \dots, A_{n-1} \subset I$, as follows.

Set $A_0 = I_1$. Since $f(I_1) \supset I_1$ i.e. $f(A_0) \supset A_0$, from our second observation, there is a subinterval $A_1 \subset A_0$ such that

$$f(A_1) = A_0 \supset A_1.$$

Then there is a subinterval $A_2 \subset A_1$, such that

$$f(A_2) = A_1 \supset A_2.$$

Note that $f^2(A_2) = A_0 = I_1$. Continuing, we find subintervals

$$A_0 \supset A_1 \supset A_2 \supset \dots \supset A_{n-3} \supset A_{n-2} \text{ such that}$$

$$f(A_{i+1}) = A_i, \quad i = 0, 1, \dots, n-3.$$

Thus $f^{n-2}(A_{n-2}) = A_0 = I_1$.

Now since $f^{n-1}(A_{n-2}) = f(I_1) \supset I_0$, there exists an interval $A_{n-1} \subset A_{n-2}$ such that

$$f^{n-1}(A_{n-1}) = I_0.$$

But $f(I_0) \supset I$, i.e. $f^n(A_{n-1}) \supset I_1$, it follows from our first observation that f^n has a fixed point p in A_{n-1} . We claim that p actually has prime period n . The reason is that the first $(n-2)$ iterations of p lie in I_1 , the $(n-1)$ th lies in I_0 , and n -th is p again. If $f^{n-1}(p)$ lies in the interior of I_0 , then it follows easily that p has prime period n . If $f^{n-1}(p)$ happens to lie on the boundary of I_0 , i.e. $f^{n-1}(p) = a$ or b , then $n = 2$ or 3 . Thus the proof is completed.

then $f(p) = a$
or $f^2(p) = a$,

$(n \geq 4)$
since the first $n-2$ iterations lie in $I_1 = [b, c]$

which is contrary to $f(A_{n-1}) = A_{n-2} \subset [b, c]$
and $f^2(A_{n-1}) = A_{n-3} \subset [b, c]$.

This theorem is just the beginning of the story. Sarkovskii's theorem is much more than that. In fact, it gives a complete accounting of which periods imply which other periods for continuous maps of R . Consider the following ordering of the natural numbers

$$\begin{aligned}
 &3 \triangleright 5 \triangleright 7 \triangleright 9 \triangleright \dots \triangleright \\
 &2 \cdot 3 \triangleright 2 \cdot 5 \triangleright 2 \cdot 7 \triangleright 2 \cdot 9 \triangleright \dots \triangleright \\
 &2^2 \cdot 3 \triangleright 2^2 \cdot 5 \triangleright 2^2 \cdot 7 \triangleright 2^2 \cdot 9 \triangleright \dots \triangleright \\
 &2^3 \cdot 3 \triangleright 2^3 \cdot 5 \triangleright 2^3 \cdot 7 \triangleright 2^3 \cdot 9 \triangleright \dots \triangleright \\
 &\dots \triangleright \\
 &\dots \cdot 2^4 \triangleright 2^3 \triangleright 2^2 \triangleright 2 \triangleright 1.
 \end{aligned}$$

That is, first list all odd numbers except one, followed by 2 times the odds, 2^2 times the odds, 2^3 times the odds, etc. This exhausts all the natural numbers with the exception of the powers of two which we list last, in decreasing order. This is the Sarkovskii ordering of the natural numbers.

Theorem 2 (Sarkovskii's theorem)

Suppose $f: R \rightarrow R$ is continuous. If f has a periodic point of prime period k , and $k \triangleright 1$ in the Sarkovskii ordering, then f also has a periodic point of prime period 1.

The proof of this full theorem does not involve new ideas but considerably more bookkeeping. It will not be given here.

Remarks:

1. If f has a periodic point whose period is not a power of two, then f must have infinitely many periodic points. Conversely, iff has only finitely many periodic points, then it necessarily has periods which are powers of two. This is confirmed in the logistic map case.

2. Sarkovskii's theorem is sharp. For instance, in the logistic map,

$$\text{when } 1 + \sqrt{6} < r < 3.54409 \dots, \quad f \text{ has a 4-cycle, but no 8-cycles.}$$

Example 1. In the logistic map, when $1 + \sqrt{8} < r < 3.8415 \dots$, f has a 3-cycle.
(3.8284...)

According to Sarkovskii's theorem, it must also have cycles of any period.

Example 2. In the logistic map, when $r = 3.628$, f has a 6-cycle.
Thus it must have infinitely many periodic cycles.

Question: Where are these infinite periodic orbits in the logistic map? Suppose $r = 3.83$, then f must have cycles of any period. Why do we see only the 3-cycle?

Answer: If a periodic orbit is repelling, then in general it can not be seen on a computer. In $r = 3.83$ case, the 3-cycle is the only attracting cycle, thus it is the only one seen on a computer. At other r values the situation is similar: at most one periodic orbit is attracting.

Question: In the logistic map, why, of all the infinitely many periodic orbits, at most one is attracting?

There is a deep answer for it.

Theorem 3 (Fatou)

Every attracting cycle for a polynomial (or a rational function) attracts at least one critical point.

For the logistic map, $f(x) = rx(1-x)$, it has only one critical point $x = \frac{1}{2}$, where $f(\frac{1}{2}) = 0$. Thus for every value of r , f has at most one attracting cycle.

The proof of the above theorem is non-trivial and is omitted here.

Comment on the orbit diagram of the logistic map:

As r continuously increases from 0 to $1 + \sqrt{8}$ (onset of the 3-cycle), periodic orbits of periods from bottom up in the Sarkovskii ordering gradually come into existence. Every time a new cycle is born, it is attracting, while the previous one becomes repelling. In this process, the periodic orbits are never destroyed. Rather they just become repelling. Thus the set of periodic orbits steadily gets larger as r increases. When $r > 1 + \sqrt{8}$, this set contains periodic orbits of any prime period.

