Universality

The amazing thing about the logistic map is that its dynamics is "universal". Many different looking maps also undergo period-doubling bifurcation to chaos, followed by periodic windows interwoven with chaotic bands, just like the logistic map!

Example 1. Plot the orbit diagram of the map

$$f(x) = rx^2(1 - x^2), \qquad 0 \le x \le 1$$

for $0 \le r \le 4$ and compare it to that of the logistic map.

Solution: The graph of this map is as shown below.

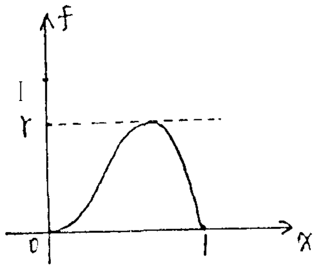
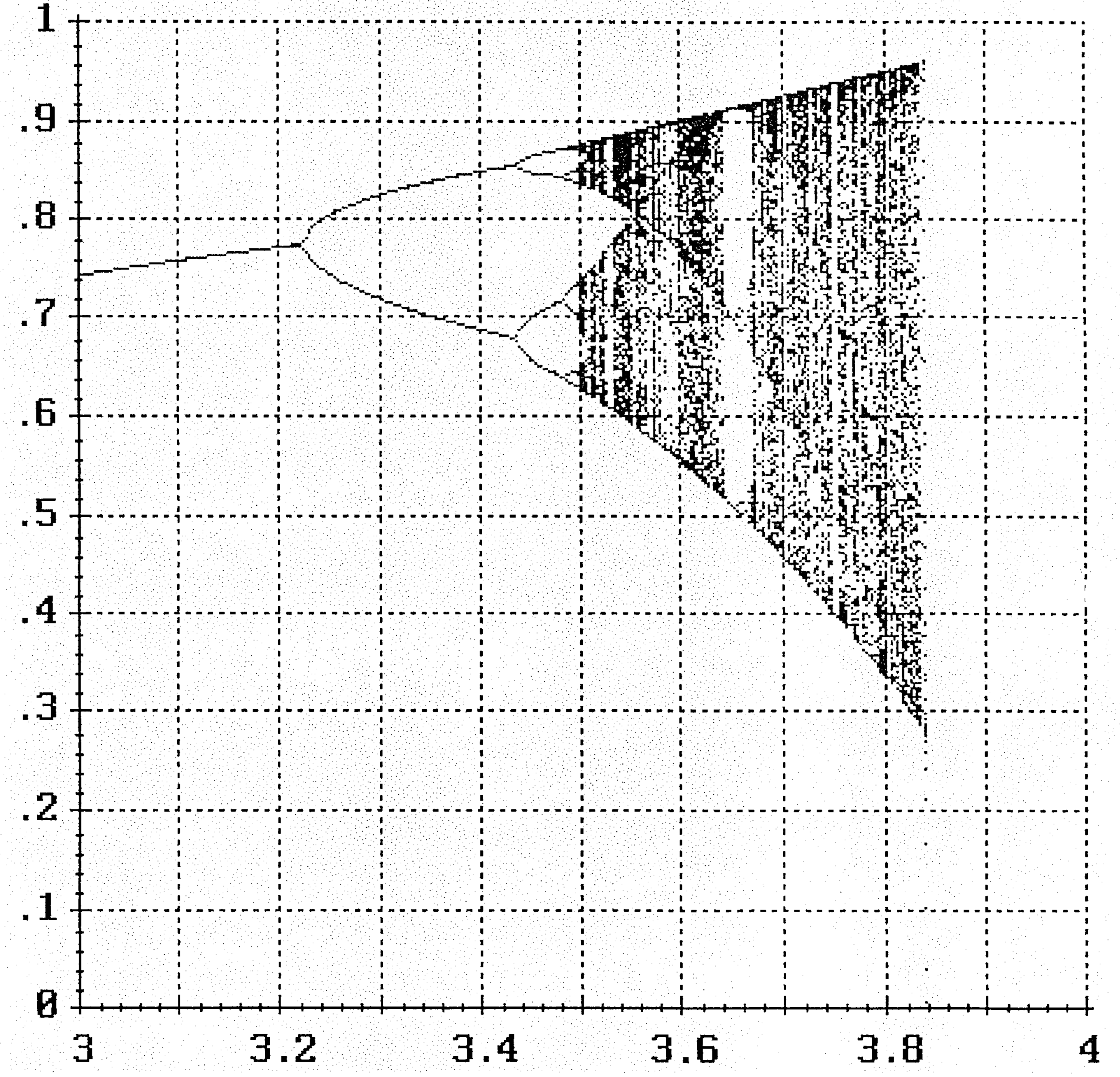


Figure 1

It is somewhat similar to the logistic map in the sense that it also has a unique maximum inside the unit interval, and $f(0) = f(1) = 0$ too. But serious differences remain. The major differerences that here $r = 0$ is a super attracting fixed point for any value of r. Other qualitative differences are also apparent.

Now let us plot the orbit diagram for this map, which is shown in Fig 2

$F(x) = rx^2(1-x^2)$



Qualitative universality, the U-sequence

The similarity of the orbit diagrams for the logistic map, the map in Example 1 and the sine map (covered in Strogatz's book) is no accident. In 1973, Metropolis, et al. proved the following result.

Theorem 1. Consider the $1-D$ map

$$x_{n+1} = r \bullet f(x_n)$$

where $0 \leq x \leq 1$ and r is a parameter

If $f(x)$ satisfies the following properties:

(1) $f(x)$ is continuous, single-valued, and piece-wise differentiable on $[0, 1]$, and strictly positive on the open unit interval, with $f(0) = f(1) = 0$;

(2) $f(x)$ has a unique maximum, $f\max$, assumed either at a point or in an interval. To the left or right of this point (or interval) $f(x)$ is strictly increasing or strictly decreasing, respectively;

(3) At any x such that $f(x) = f\max$, the derivative exists and is equal to zero;

(4) Let $r_{\max} = \frac{1}{f_{\max}}$. Then there exists a r_0 such that, for $r_0 < r < r_{\max}$, $rf(x)$ has only two fixed points, the origin and $x_f(r)$, say, both of which are repelling;

then, as r varies from r_0 to $r_{\max}$, an infinite sequence of attracting periodic orbits (now called the U-sequence), always in the same order, can be constructed for this map $rf(x)$, regardless of its details. Up to period 6, the U-sequence is

2,4,6,5,3,6,5,6,4,6,5,6.

Examples of this class of maps:

1. $x_{n+1} = rx_n(1-x_n)$, $3 < r < 4$ (the logistic map);

2. $x_{n+1} = r\sin\pi x_n$, $r_0 < r < 1$ (with $0.71 < r_0 < 0.72$) (the sine map).

Note: the map in example 1: $x_{n+1} = rx_n^2(1-x_n^2)$ does not belong to this class because condition **(4)** does not hold.

Comments:

1. Generally speaking, this U-sequence of attracting periodic orbits are not the only possible orbits in a map of this class. Nevertheless they constitute perhaps the most interesting family of attracting periodic orbits in virtue of the universality of their structure and of their order of occurrence.

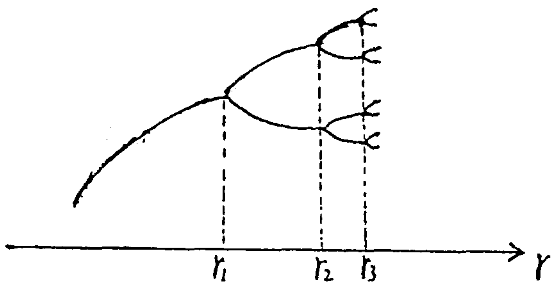
2. The four conditions in theorem 1 are not the necessary conditions for a map $rf(x)$ to exhibit the U-sequence. For instance, the map $x_{n+1} = rx_n^2(1-x_n^2)$ does not satisfy condition (4), yet it also exhibits the U-sequence.

3. The U-sequence is qualitative: it dictates order, but not the precise parameter values at which attracting periodic orbits occur.

<u>Quantitative universality</u>

What is more surprising is the quantitative universality of the period-doubling bifurcation in the logistic map. In that map, the r values at whch period-doubling bifurcations occur are:

$r_1 = 3$	period 2 is born	$\Delta_n \equiv r_n - r_{n-1}$	Δ_n/Δ_{n+1}
$r_2 = 3.449489....$	4	$\approx 4.4949 \bullet 10^{-1}$	
$r_3 = 3.544090....$	8	$\approx 9.4611 \bullet 10^{-2}$	4.7514...
$r_4 = 3.564407....$	16	$\approx 2.0316 \bullet 10^{-2}$	4.6562...
$r_5 = 3.568759....$	32	$\approx 4.3521 \bullet 10^{-3}$	4.6682...
$r_6 = 3.569692....$	64	$\approx 9.3219 \bullet 10^{-4}$	4.6687...
$r_7 = 3.569891....$	128	$\approx 1.9964 \bullet 10^{-4}$	4.6693...
$r_\infty = 3.569946$	∞		4.6692...

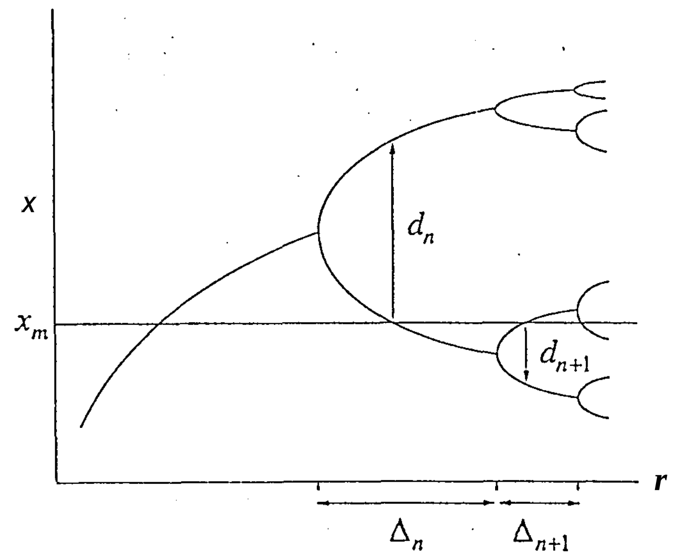


Note that the successive bifurcations come faster and faster, and as $n \to \infty$, r_n converges to r_∞ geometrically with

$$\delta = \lim_{n\to\infty} \text{——} = \lim_{n\to\infty} \frac{\Delta_n}{\Delta_{n+1}} = 4.6692016091029....$$

What is surprising is that this δ-value is actually universal to all maps $rf(x)$ where $f(x)$ has a unique differentiable maximum, i.e. such maps undergoing period-doubling bifurcations possess the same value for 6, independent of the details of these maps. This fact was first discovered by Mitchell Feigenbaum in 1975 and proved by him using the renormalization technique in 1978.

Another constant related to scaling in the x-direction is also universal.



Let x_m denote the maximum off, and let d_n denote the distance from x_m to the nearest point in a 2^n-cycle, then the ratio 'd_n/d_{n+1} also tends to a universal limit as $n \to \infty$:

$$\alpha = \lim_{n\to\infty} \frac{d_n}{d_{n+1}} = -2.5029078750957....$$

independent of the precise form off; as long as f has a unique differentiable maximum and f'' does not vanish there.

What do $1-D$ maps have to do with science?

Most physical systems are governed by differential equations (ordinary or partial), not $1-D$ maps. So how could the results of $1-D$ maps be relevant in such systems? I will use one example to answer this question.

Consider a system of 0.D.E.s (Rossler system):

$$\begin{cases} dx/dt = -y - z \\ dy/dt = x + ay \\ dz/dt = b + z(x-c) \end{cases}$$

where a, b and c are parameters. For simplicity, we keep $a = b = 0.2$ fixed, and let c vary. The figure below shows the projections of the system's attractor onto the (x,y) plane for different values of c.

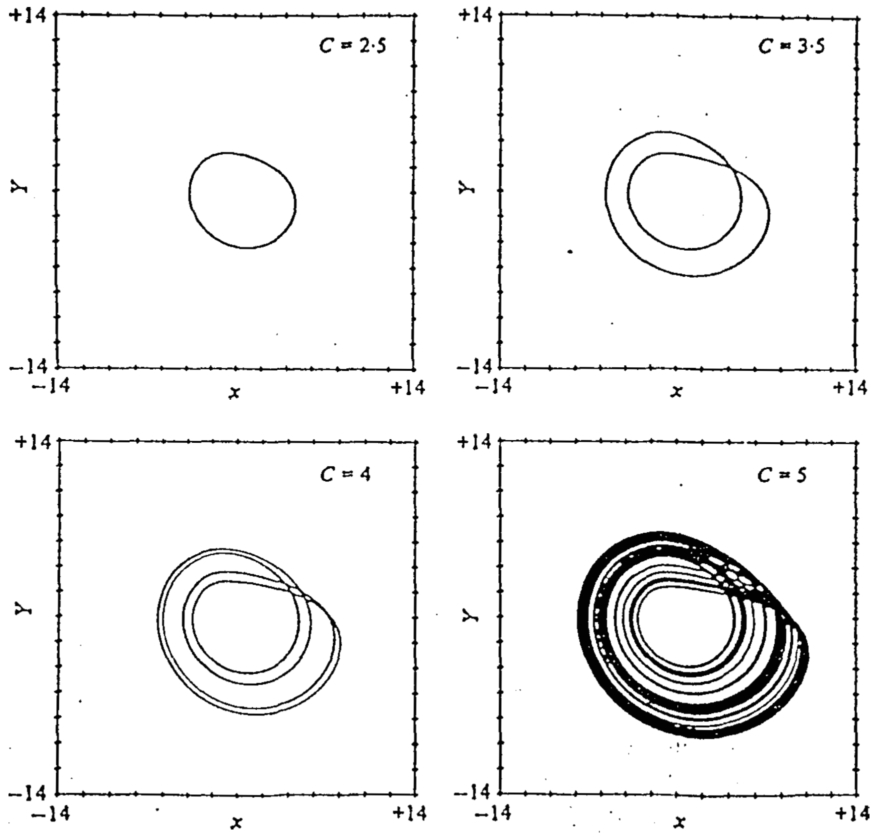
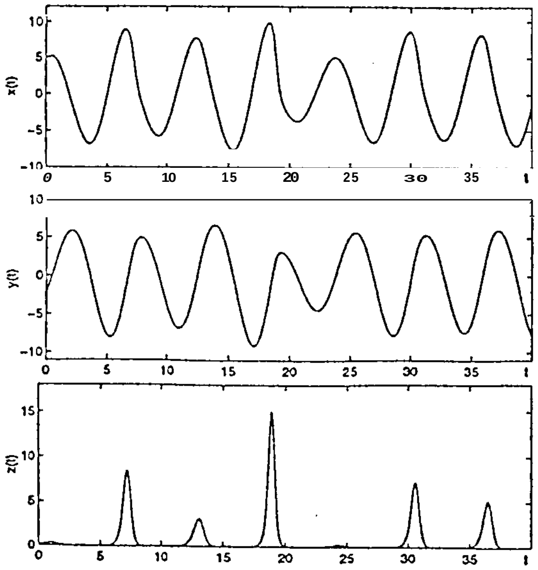


Figure 10.6.6 Olsen **and** Degn *(1985).* p. **185**

At $c = 2.5$, the attractor is a simple limit cycle. As c is increased to 3.5, the limit cycle goes around twice before closing, and its period is approximately twice that of the original cycle. This is what period-doubling bifurcation looks like in a continuous-time system. At $c = 4$, the period is doubled again. After an infinite cascade of further period-doubling, one obtains a strange attractor shown at $c = 5$.

The above story sounds very familiar. But how can we make a connection between the Rossler system and the $1-D$ map? Here we use a trick invented by Ed Lorenz. The idea is to extrat a $1-D$ map from the continuous solutions of the system. For a given value of c, we record the successive local maxima of $x(t)$ on the attractor. Then we plot x_{n+1} vs. x_n, where x_n denotes the ***nth*** local maximum. For $c = 5$, a solution $x(t)$ on the strange attractor is shown in the left figure below, and this Lorenz map is shown in the right figure. Surprisingly all the data points fall very nearly on a one-dimensional curve which looks much like the logistic map!

Solutions of the Rossler system on the attractor ($c = 5$).



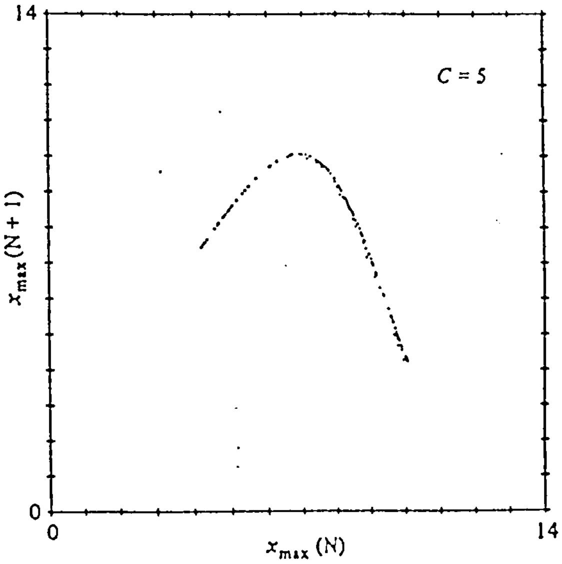


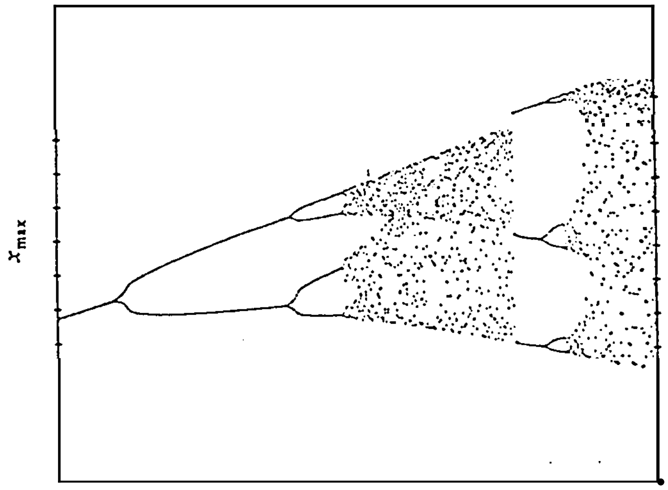
Figure 10.6.7 Olsen and Degn **(1985),p. 186**

Hence we make the following connection. Let

$$x_{n+1} = F(x_n; c),$$

where the function F is the Lorenz map. Note that limit cycles of the Rossler system correspond to attracting periodic orbits of the Lorenz map, and strange attractors of the Rossler system correspond to chaotic bands of the Lorenz map.

Since the Lorenz map is unimodal, we can now predict the U-sequence and quantitative period-doubling bifurcations in the Rossler system. Indeed, when all the local maxima x_n on the attractor of the Rossler system are plotted for each c, the following orbit diagram will be obtained (Olsen and Degn 1985). Not surprisingly, we see period-doubling bifurcation to chaos, followed by periodic windows in the chaotic band. The period-3 window is large and clearly visible.



Generally speaking, if the Lorenz map of a certain physical system is nearly one-dimensional and unimodal, then the universality theory applies. This is certainly the case for the Rossler system. But not all systems have one-dimensional Lorenz maps. For the Lorenz map to be nearly one-dimensional, the strange attractor has to be very flat, i.e. its Hausdorff dimension has to be slightly larger than two. This requires the system to be highly dissipative, and only two or three degrees of freedom are truly active, and the rest follow along passively.

Experimental tests

Since Feigenbaum's work, sequences of period-doubling bifurcations have been reported in a variety of experimental systems. The following table summarizes a few experiments on fluid convection and nonlinear electronic circuits. The experimental estimates of δ are shown along with the errors quoted by the experimentalists. Note: **4.3(8)** means 4.3 ± 0.8, etc.

Experiment	Number of period doublings	δ	Authors
Hydrodynamic			
water	4	4.3(8)	Giglio et al. (1981)
mercury	4	4.4(1)	Libchaber et al. (1952)
Electronic			
diode	4	4.5(6)	Linsay (1981)
diode	5	4.3 (1)	Testa et al. (1982)
transistor	4	4.7 (3)	Arecchi and Lisi (1952)
Josephson simul.	3	4.5 (3)	Yeh and Kao (1982)

Refer to those papers for details.

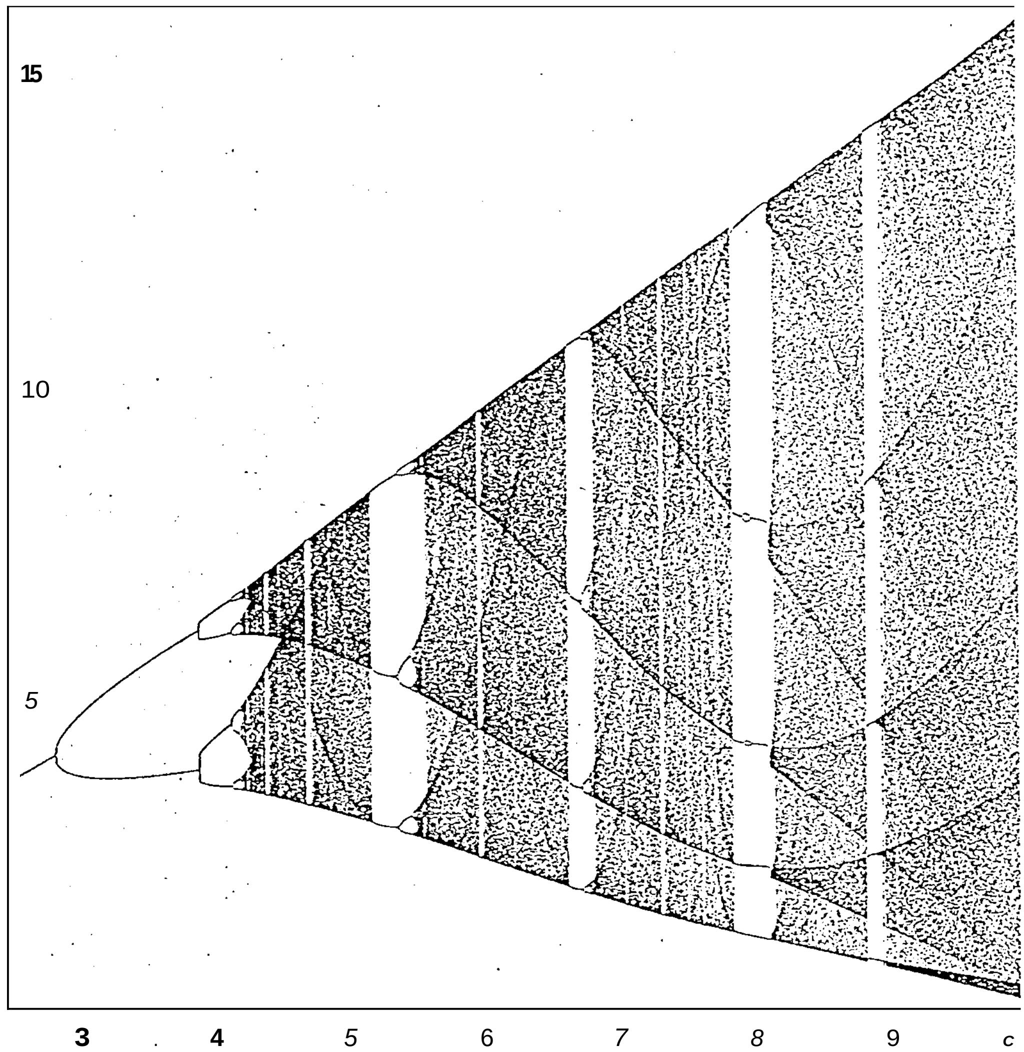


Figure 12.30 : Feigenbaum diagram for Rossler's System. The plot shows the parameter range $2.5 < c < 10.0$. Vertically the absolute values of the minimal x-values of the corresponding trajectories from the attractor are shown. This corresponds to a projection of a Lorenz map diagram such as given in figure 12.28 on the vertical axis. Initially, for small values of c. the attractor consists of a periodic orbit which has only one local minimum of 2-values, i.e., it is **a** single loop. **As** the parameter c increases, this periodic orbit undergoes a period-doubling bifurcation. Check that for $c = 8$ there are 5 points in the diagram corresponding to the periodic solution shown in figure 12.29 which has 5 loops.